Attempt all questions and show all your work. Due March 24, 2010 in class, by the end of class with a signed honesty declaration.

1. For each function, find (a) the critical numbers, (b) the open intervals where the function is increasing, and (c) the open intervals where the function is decreasing.

   (a) $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$

   Solution: $f'(x) = 2x^2 - 2x - 4 = 2(x^2 - x - 2) = 2(x-2)(x+1)$
   Domain of f: $\mathbb{R}$
   Critical numbers: 2, -1
   The function is increasing on: $(-\infty, -1) \cup (2, \infty)$, and decreasing on: $(-1, 2)$.

   (b) $f(x) = x\sqrt{9 - x^2}$

   Solution: $f'(x) = \frac{9-2x^2}{\sqrt{3-x}\sqrt{x+3}}$
   Domain of f: $[-3, 3]$
   Critical numbers: $-3, 3, \frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}$
   The function is increasing on: $(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, and decreasing on: $(-3, \frac{-3}{\sqrt{2}}) \cup (\frac{3}{\sqrt{2}}, 3)$.

   (c) $f(x) = \ln \frac{5x^2+4}{x^2+1}$

   Solution: $f'(x) = \frac{2x}{(5x^2+4)(x^2+1)}$
   Domain of f: $\mathbb{R}$
   Critical numbers: 0
   The function is increasing on: $(0, \infty)$, and decreasing on: $(-\infty, 0)$.

   (d) $f(x) = x2^{-x^2}$

   Solution: $f'(x) = \frac{1-2\ln 2x^2}{2^{x^2}}$
   Domain of f: $\mathbb{R}$
   Critical numbers: $-\frac{1}{\sqrt{\ln 4}}, \frac{1}{\sqrt{\ln 4}}$
   The function is increasing on: $(-\frac{1}{\sqrt{\ln 4}}, \frac{1}{\sqrt{\ln 4}})$, and decreasing on: $(-\infty, -\frac{1}{\sqrt{\ln 4}}) \cup (\frac{1}{\sqrt{\ln 4}}, \infty)$.

2. Find the x-value of all points where the functions defined as follows have any local (relative) extrema. Find the value(s) of any local (relative) extrema as well.

   (a) $f(x) = x^2 + \frac{1}{x}$

   Solution: $f'(x) = \frac{2x^3-1}{x^2}$
   Domain of f: $(-\infty, 0) \cup (0, \infty)$

Critical no's: $x = \frac{1}{\sqrt[3]{2}} = 2^{-1/3}$

Local mins: $(2^{-1/3}, 2^{-2/3} + 2^{1/3})$ Local max: None

(b) $f(x) = \frac{x^2-6x+9}{x+2}$

Solution: $f'(x) = \frac{(x+7)(x-3)}{(x+2)^2}$

Domain of f: $(-\infty, -2) \cup (-2, \infty)$

Critical no's: $-7, 3$

Local mins: $(3, 0)$ Local max: $(-7, -20)$

(c) $f(x) = 3xe^x + 2$

Solution: $f'(x) = (3x + 3)e^x$

Domain of f: $\mathbb{R}$

Critical no's: -1

Local mins: $(-1, \frac{-3}{e} + 2)$ Local max: None

(d) $f(x) = \frac{x^2}{\ln x}$

Solution: $f'(x) = \frac{2x \ln x - x}{\ln^2 x}$

Domain of f: $(0, 1) \cup (1, \infty)$

Critical no's: $\sqrt{e}$

Local mins: $(\sqrt{e}, 2e)$ Local max: None

3. Find $f''(x)$ for each function, and find the open intervals on which the function is concave upward or concave downwards. Find any inflection points.

(a) $f(x) = -x^4 + 7x^3 - \frac{x^2}{2}$

Solution: $f'(x) = -4x^3 + 21x^2 - x$ $f''(x) = -12x^2 + 42x - 1$

$f''(x) = 0$ if and only if $x = x = -\frac{\sqrt{429}-21}{12}$, $x = \frac{\sqrt{429}+21}{12}$

f is concave up on $(-\frac{\sqrt{429}-21}{12}, \frac{\sqrt{429}+21}{12})$ f is concave down on $(-\infty, -\frac{\sqrt{429}-21}{12}) \cup (\frac{\sqrt{429}+21}{12}, \infty)$

(NOTE: There must have been a typo in the question–this shouldn't have been so nasty!)

(b) $f(x) = \ln x + \frac{1}{x}$

Solution: Domain of f: $(0, \infty)$.

$f'(x) = \frac{x-1}{x^2}$ $f''(x) = -\frac{x-2}{x^3}$

$f''(x) = 0$ if and only if $x = 2$

f is concave up on $(0, 2)$ f is concave down on $(2, \infty)$

Inflection Point: $(2, \ln 2 + \frac{1}{2})$

(c) $f(x) = -x(x-3)^2$

Solution: $f'(x) = -3x^2 + 12x - 9$ $f''(x) = 12 - 6x$
$f''(x) = 0$ if and only if $x = 2$
f is concave up on $(-\infty, 2)$ f is concave down on $(2, \infty)$
Inflection point: $(2, -2)$

(d) $f(x) = x^2 + 8\ln|x+1|$

Solution: Domain: $(-\infty, -1) \cup (-1, \infty)$
$f'(x) = \frac{2x^2+2x+8}{x+1}$ $f''(x) = \frac{2x^2+4x-6}{x^2+2x+1} = \frac{2(x+3)(x-1)}{(x+1)^2}$
$f''(x) = 0$ if and only if $x = -3, 1$
f is concave up on $(-\infty, -3) \cup (1, \infty)$ f is concave down on $(-3, -1) \cup (-1, 1)$
Inflection points: $(-3, 9 + 8\ln(2))$, $(1, 1 + 8\ln(2))$

4. Consider the curve given by the function

$$f(x) = \frac{-2x}{x^2 - 4}.$$

Note then that

$$f'(x) = \frac{2x^2 + 8}{(x^2 - 4)^2} \qquad \text{and} \qquad f''(x) = \frac{-4x^3 - 48x}{(x^2 - 4)^3}.$$

(a) Determine the domain of $f(x)$ and its x-intercepts.

Solution:

$$f(x) = \frac{-2x}{x^2 - 4} = \frac{-2x}{(x-2)(x+2)}$$

Domain of f: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

(b) Find all vertical asymptotes of $f(x)$. (Show all your work).

Solution:

$$\lim_{x \to -2^-} f(x) = \frac{4}{-4(\text{small -ve})} = \infty$$
$$\lim_{x \to -2^+} f(x) = \frac{4}{-4(\text{small +ve})} = -\infty$$
$$\lim_{x \to 2^-} f(x) = \frac{-4}{4(\text{small -ve})} = \infty$$
$$\lim_{x \to 2^+} f(x) = \frac{-4}{4(\text{small +ve})} = -\infty$$

(c) Find all critical points of $f(x)$ (that is, all critical numbers, together with their y values).

Solution: $f'(x) = \frac{2x^2+8}{(x^2-4)^2} = 0$ never happens. Thus there are no critical numbers.

(d) Find the open intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

Solution: Increasing: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ (the entire domain). Decreasing: Never

(e) Find the coordinates of points at which the local maxima and/or local minima occur.

Solution: No local mins or maxs.

(f) Determine the open intervals upon which $f(x)$ is concave up and the open intervals where $f(x)$ is concave down.

Solution: $f''(x) = \frac{-4x^3-48x}{(x^2-4)^3} = 0$ when $-4x^3 - 48x = 0$, when $-4x(x^2 + 12) = 0$, which only happens when $x = 0$.

f is concave up on $(-\infty, -2) \cup (0, 2)$, and concave down on $(-2, 0) \cup (2, \infty)$.

(g) Find the coordinates of the points of inflection.

Solution: $(0, 0)$.

(h) Use the above information to give a neat sketch of the graph $y = f(x)$.

Solution: