MATH 2300 Assignment 5 10R

Attempt all questions and show all your work. Due April 9, 2010. Hand in your paper in the box outside my office door (425 Machray Hall) **BEFORE 1:30pm**.

1. For each of the following matrices, find the characteristic polynomial, the eigenvalues, and the corresponding eigenspaces for each eigenvalue:
   (a) $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$
   (c) $\begin{bmatrix} 4 & 2 & -2 & 2 \\ 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -3 & 5 \end{bmatrix}$
2. What are the eigenvalues and eigenvectors of I_n?
3. Compute A^9 using the process of diagonalization if $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$.
4. Show that $\begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ is not diagonalizable.
5. Consider the vector space P_n of polynomials of degree less than or equal to n. Let f and g be two elements of P_n.
   (a) Prove that
   $$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$$
   defines an inner product on P_n (note this is defined on P_n where n is arbitrary, and thus you must prove this in general).
   (b) Find $\langle x^2 + 2x - 1, 4x + 1 \rangle$
   (c) Show that $f(x) = 1$ and $g(x) = \frac{1}{2} - x$ are orthogonal
   (d) Determine which of the functions $g(x) = x^2 + 2x - 3$ or $h(x) = x^2 - 3x + 4$ is closest to the function $f(x) = x^2$.
6. Show that in any inner product space V, for all $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.
7. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Let V be the set of all eigenvectors corresponding to λ, together with the zero vector. Prove that V is a subspace of $\mathbb{R}^n$.
8. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.
9. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

10. An $n \times n$ matrix A is said to be **nilpotent** if for some $k \in \mathbb{Z}^+$, A^k is a zero matrix. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

11. Prove that if $B = C^{-1}AC$, then B and A have the same eigenvalues (HINT: Look at the characteristic polynomials of B and A).

12. Let $\mathbf{v}$ be a nonzero vector in an inner product space V. Let W be the set of all vectors in V that are orthogonal to $\mathbf{v}$. Prove that W is a subspace of V.