MATH 2300 Sample Proofs

This document contains a number of theorems, the proofs of which are at a difficulty level where they could be put on a test or exam. This should not be taken as an indication that the only theorems on tests or exams will be taken from this document, nor that every (or any) theorem in this document need be tested. This document is for information purposes only. Questions marked with a * are a little harder than the others, and the more stars, the harder the question, but **all are testable**. More solutions will be provided as time allows.

VECTOR SPACE PROOFS

1. Prove that for any set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in a vector space V, $\text{span}(S)$ is a subspace of V.

Solution: Let $\mathbf{u}, \mathbf{w} \in \text{span}(S)$, $k \in \mathbb{R}$. Then there exist $c_1, \dots c_n \in \mathbb{R}$ and $k_1, \dots, k_n \in \mathbb{R}$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$$

and

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n.$$

A1)

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n) + (k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \dots + k_n\mathbf{v}_n) \\ &= (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + \dots + (c_n + k_n)\mathbf{v}_n \in \text{span}(S). \end{aligned}$$

Thus $\text{span}(S)$ is closed under addition.

M1)

$$\begin{aligned} k\mathbf{u} &= k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n) \\ &= k(c_1\mathbf{v}_1) + k(c_2\mathbf{v}_2) + \dots + k(c_n\mathbf{v}_n) \\ &= (kc_1)\mathbf{v}_1 + (kc_2)\mathbf{v}_2 + \dots + (kc_n)\mathbf{v}_n \in \text{span}(S). \end{aligned}$$

Thus $\text{span}(S)$ is closed under scalar multiplication.

Thus by the subspace theorem, $\text{span}(S)$ is a subspace of V.

2. Prove that if S is a linearly independent set of vectors, then S is a basis for $\text{span}(S)$.

Solution: To be a basis for $\text{span}(S)$, it must be linearly independent and span the space. Certainly the span of S is equal to the entire space, $\text{span}(S)$ (by definition), and S is given to be linearly independent in the question. Thus S is a basis for $\text{span}(S)$.

3. Show that if A is an $m \times n$ matrix, then the solution set V to the equation $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

Solution: A1) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ be two solutions to the equation $A\mathbf{x} = \mathbf{0}$ (that is, $\mathbf{x}_1, \mathbf{x}_2 \in V$). Then $\mathbf{x}_1 + \mathbf{x}_2 \in \mathbb{R}^n$, and

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

Thus $\mathbf{x}_1 + \mathbf{x}_2 \in V$.

M1). Let $\mathbf{x}_1 \in V$, $k \in \mathbb{R}$. Then $k\mathbf{x}_1 \in \mathbb{R}^n$, and

$$\begin{aligned} A(k\mathbf{x}_1) &= k(A\mathbf{x}_1) \\ &= k(\mathbf{0}) = \mathbf{0}. \end{aligned}$$

Thus $k\mathbf{x}_1 \in V$ as well.

Thus by the subspace theorem, V is a subspace of $\mathbb{R}^n$.

4. Prove that any finite set of vectors containing the zero vector is linearly dependent.

Solution: Let $S = \{\mathbf{0}, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then the equation

$$c_1\mathbf{0} + c_2\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

has the solution $c_1 = 1, c_2 = c_3 = \cdots = c_n = 0$, which is not all zeros, and thus the set S is linearly dependent.

5. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v} \in V$ can be expressed as a linear combination of the elements of S in exactly one way.

Solution: Let $\mathbf{v} \in V$. Let $c_1, \ldots, c_n, k_1, \ldots, k_n \in \mathbb{R}$ be such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

and

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n.$$

Then subtracting straight down we get

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \ldots + (c_n - k_n)\mathbf{v}_n.$$

But since S is a basis, S is linearly independent, and thus the only way this could happen is if $c_1 - k_1 = 0$, ..., $c_n - k_n = 0$. Thus $c_1 = k_1, \ldots, c_n = k_n$ and so $\mathbf{v}$ can only be written as a linear combination of the elements of S in one way.

*6. Given that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in some vector space V, prove that:

(a) the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

(b) the set $\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$ is linearly independent.

(c) the set $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}\}$ is linearly independent.

*7. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be such that $\mathbf{u} \bullet \mathbf{v} = 0$. Prove that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set.

*8. Let V be a vector space. Prove that for every $\mathbf{u} \in V$, $0 \cdot \mathbf{u} = \mathbf{0}$.

Solution: Let $\mathbf{u} \in V$. Then

$$\begin{aligned} 0 \cdot \mathbf{u} &= (0+0) \cdot \mathbf{u} && \text{(since } 0+0=0\text{)} \\ &= 0 \cdot \mathbf{u} \oplus 0 \cdot \mathbf{u}. && \text{(axiom } M3\text{)} \end{aligned}$$

By axiom $A5$, there exists $-(0 \cdot \mathbf{u})$. Then,

$$0 \cdot \mathbf{u} \oplus -(0 \cdot \mathbf{u}) = \mathbf{0} \qquad \text{(by } A5\text{)},$$

and

$$\begin{aligned} 0 \cdot \mathbf{u} \oplus -(0 \cdot \mathbf{u}) &= (0 \cdot \mathbf{u} \oplus 0 \cdot \mathbf{u}) \oplus -(0 \cdot \mathbf{u}) && \text{(by above)} \\ &= 0 \cdot \mathbf{u} \oplus (0 \cdot \mathbf{u} \oplus -(0 \cdot \mathbf{u})) && \text{(by } A3\text{)} \\ &= 0 \cdot \mathbf{u} \oplus \mathbf{0} && \text{(by } A5\text{)} \\ &= 0 \cdot \mathbf{u}. && \text{(by } A4\text{)} \end{aligned}$$

Therefore

$$0 \cdot \mathbf{u} = \mathbf{0}.$$

*9. Let V be a vector space. Prove that for every $k \in \mathbb{R}$, $k \cdot \mathbf{0} = \mathbf{0}$.

Solution: Let $k \in \mathbb{R}$. Then

$$\begin{aligned} k \cdot \mathbf{0} &= k \cdot (\mathbf{0} \oplus \mathbf{0}) && \text{(by } A4\text{)} \\ &= k \cdot \mathbf{0} \oplus k \cdot \mathbf{0} && \text{(by } M2\text{).} \end{aligned}$$

By $A5$, there exists $-(k \cdot \mathbf{0})$. Then,

$$k \cdot \mathbf{0} \oplus -(k \cdot \mathbf{0}) = \mathbf{0}, \qquad \text{(by } A5\text{)}$$

but

$$\begin{aligned} k \cdot \mathbf{0} \oplus -(k \cdot \mathbf{0}) &= (k \cdot \mathbf{0} \oplus k \cdot \mathbf{0}) \oplus -(k \cdot \mathbf{0}) && \text{(by above)} \\ &= k \cdot \mathbf{0} \oplus (k \cdot \mathbf{0} \oplus -(k \cdot \mathbf{0})) && \text{(by } A3\text{)} \\ &= k \cdot \mathbf{0} \oplus \mathbf{0} && \text{(by } A5\text{)} \\ &= k \cdot \mathbf{0}. && \text{(by } A4\text{)} \end{aligned}$$

Therefore

$$k \cdot \mathbf{0} = \mathbf{0}.$$

*10. Let V be a vector space. Prove that for every $\mathbf{u} \in V$, $(-1) \cdot \mathbf{u} = -\mathbf{u}$.

Solution: Let $\mathbf{u} \in V$. Then,

$$\begin{aligned} (-1) \cdot \mathbf{u} \oplus \mathbf{u} &= (-1) \cdot \mathbf{u} \oplus 1 \cdot \mathbf{u} && \text{(by } M5\text{)} \\ &= (-1 + 1) \cdot \mathbf{u} && \text{(by } M3\text{)} \\ &= 0 \cdot \mathbf{u} \\ &= \mathbf{0} && \text{(by part 1).} \end{aligned}$$

Therefore, $-\mathbf{u} = (-1) \cdot \mathbf{u}$.

*11. Let V be a vector space. Prove that if for some $k \in \mathbb{R}$ and $\mathbf{u} \in V$, $k \cdot \mathbf{u} = 0$, then either $k = 0$, or $\mathbf{u} = 0$.

Solution: Let $k \in \mathbb{R}$, $\mathbf{u} \in V$, and assume that $k \cdot \mathbf{u} = \mathbf{0}$. If $k = 0$, then this is true, by part 1, so assume that $k \neq 0$. Then

$$\begin{aligned} \frac{1}{k} \cdot (k \cdot \mathbf{u} &= (\frac{1}{k}k) \cdot \mathbf{u}) && \text{(by } M4\text{)} \\ &= 1 \cdot \mathbf{u} \\ &= \mathbf{u} && \text{(by } M5\text{),} \end{aligned}$$

but

$$\begin{aligned} \frac{1}{k} \cdot (k \cdot \mathbf{u}) &= \frac{1}{k} \cdot \mathbf{0} && \text{(by assumption)} \\ &= \mathbf{0} && \text{(by part 2).} \end{aligned}$$

Therefore $\mathbf{u} = \mathbf{0}$.

*12. Prove that a set of vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others.

*13. Let A be a $m \times n$ matrix. Prove that if both the set of rows of A and the set of columns of A form linearly independent sets, then A must be square.

Solution: Let $r_1, \dots, r_m \in \mathbb{R}^n$ be the rows of A and let $c_1, \dots, c_n \in \mathbb{R}^m$ be the columns of A. Since the set of rows is linearly independent, and the rows are elements of $\mathbb{R}^n$, it must be that $m \leq n$. Similarly, since the set of columns is linearly independent, and the columns are elements of $\mathbb{R}^m$, it must be that $n \leq m$. Thus $m = n$.

**14. Let V be the set of 2×2 matrices, together with the operation $\oplus$ defined for any 2×2 matrices A and B as

$$A \oplus B = AB \text{ (the usual matrix multiplication)},$$

and with the standard scalar multiplication for matrices.

(a) Show that the vector space axiom $A4$ holds.

(b) Prove that V is **not** a vector space.

**15. Let

$$V = \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0\}$$

together with the operations defined as follows: for $(a, b), (c, d) \in V$, $k \in \mathbb{R}$,

$$(a, b) \oplus (c, d) = (ac, bd)$$

$$k \cdot (a, b) = (a^k, b^k).$$

(a) Show that the vector space axiom $M3$ holds in this space.

(b) Does the axiom $A4$ hold in this space? If so, find the zero vector and prove it is the zero vector. If not, show that there is no possible zero vector.

**16. Let V be a vector space, and let W_1 and W_2 be subspaces of V. Prove that the set

$$U = \{\mathbf{v} : \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}$$

(that is, U is the set of vectors in BOTH W_1 and W_2). Prove that U is a subspace of V as well.

Solution: A1) Let $\mathbf{u}, \mathbf{v} \in U$. Then $\mathbf{u}, \mathbf{v} \in W_1$ and $\mathbf{u}, \mathbf{v} \in W_2$. Then since $A1$ holds in W_1, $\mathbf{u} + \mathbf{v} \in W_1$, and since $A1$ holds in W_2, $\mathbf{u} + \mathbf{v} \in W_2$ as well. Thus $\mathbf{u} + \mathbf{v} \in U$ and so $A1$ holds.

M1) Let $\mathbf{u} \in U$, and let $k \in \mathbb{R}$. Then $\mathbf{u} \in W_1$ and $\mathbf{u} \in W_2$, and so since $M1$ holds in W_1, $k\mathbf{u} \in W_1$, and since $M1$ holds in W_2, $k\mathbf{u} \in W_2$ as well. Thus $k\mathbf{u} \in U$, and so $M1$ holds.

Thus, but the subspace theorem, U is a subspace of V.

**17. Let W be a subspace of a vector space V, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in W$. Prove then that every linear combination of these vectors is also in W.

Solution: Let $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since W is a subspace (and thus a vector space), since W is closed under scalar multiplication ($M1$), we know that $c_1\mathbf{v}_1, c_2\mathbf{v}_2$, and $c_3\mathbf{v}_3$ are all in W as well. Then since W is closed under addition (A1), we know that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is also in W. Then applying closure under addition (A1) again, we get that

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \in W.$$

**18. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Solution: Assume $r > n$, and assume that for each i, $1 \le i \le r$,

$$\mathbf{v}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n}).$$

Let $c_1, c_2, \ldots, c_r \in \mathbb{R}$ be such that

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r = \mathbf{0}.$$

This produces the homogeneous system of equations:

$$\begin{bmatrix} v_{1,1} & v_{2,1} & \cdots & v_{r,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{r,2} \\ \vdots & \vdots & & \vdots \\ v_{1,n} & v_{2,n} & \cdots & v_{r,n} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The coefficient matrix of this system has n rows and r columns. But $r > n$. Therefore this system is guaranteed to have a parameter, and since it is a homogeneous system and has at least one solution, it therefore has infinitely many solutions. Thus there is at least one solution for the c_i's that is not all zero, and so the set S is linearly dependent.

19. Prove that the range of a linear transformation $T : V \to W$ is a subspace of W.

20. Prove that given two linear transformations $T_1 : U \to V$ and $T_2 : V \to W$, the composition $T_2 \circ T_1 : U \to W$ is also a linear transformation.

21. Prove that for any linear transformation $T : V \to W$, $\ker(T)$ is a subspace of W.

22. Prove that If $T_1 : U \to V$ is one-to-one, and $T_2 : V \to W$ is one-to-one, then the composition $T_2 \circ T_1 : U \to W$ is also one-to-one.

23. If $T_1 : U \to V$ is onto, and $T_2 : V \to W$ is onto, then the composition $T_2 \circ T_1 : U \to W$ is also onto.

24. Prove that for any one-to-one linear transformation $T : V \to W$, T^{-1} is also a one-to-one linear transformation.

25. Prove that for any $m \times n$ matrix M, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation.

*26. If $T : V \to W$ is a linear transformation, then prove each of the following:

- If T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$.
- If $\ker(T) = \{\mathbf{0}\}$, then T is one-to-one.

*27. If V is a finite-dimensional vector space, and $T : V \to V$ is a linear operator, then prove that if T is one-to-one, then the range of T is all of V.

*28. Prove that if $T : V \to W$ is an isomorphism between V and W (one-to-one and onto), and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.

*29. Prove that every vector space of dimension n is isomorphic to $\mathbb{R}^n$.

**30. Let $T : V \to W$ be a one-to-one linear transformation. Prove that if $\dim(V) = \dim(W)$ (and both V and W are finite-dimensional), then T is an isomorphism.

Solution: Assume $\dim(V) = \dim(W)$ and that $T : V \to W$ is one-to-one. Assume further (in hopes of a contradiction) that T is not onto. Then there exists $\mathbf{w} \in W$ such that $\mathbf{w} \notin T(V)$. But then since $T(V)$ is a subspace of W, but we know that

$T(V) \neq W$, it follows that $\operatorname{rank}(T) = \dim(T(V)) < \dim(W)$. But then, by the Dimension Theorem,

$$\begin{aligned} \operatorname{nullity}(T) &= \dim(V) - \operatorname{rank}(T) \\ &= \dim(V) - \dim(T(V)) \\ &> \dim(V) - \dim(W) \\ &= \dim(V) - \dim(V) \\ &= 0. \end{aligned}$$

Then since $\operatorname{nullity}(T) > 0$, $\ker(T) \neq \{\mathbf{0}\}$, and thus there are two vectors in V that map to the zero vector in W. Thus T is not 1:1, a contradiction.

**31. Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations. Prove that if $T_2 \circ T_1$ is one-to-one, then T_1 must be one-to-one.

Solution: Suppose that $T_2 \circ T_1 : U \to W$ is one-to-one. That is, for all $\mathbf{u}, \mathbf{v} \in U$, if $(T_2 \circ T_1)(\mathbf{u}) = (T_2 \circ T_1)(\mathbf{v})$, then $\mathbf{u} = \mathbf{v}$.

It remains to show that $T_1 : U \to V$ is one-to-one. Let $\mathbf{u}, \mathbf{v} \in U$ and assume that $T_1(\mathbf{u}) = T_1(\mathbf{v})$. Then certainly

$$T_2(T_1(\mathbf{u})) = T_2(T_1(\mathbf{v})),$$

and therefore,

$$(T_2 \circ T_1)(\mathbf{u}) = (T_2 \circ T_1)(\mathbf{v}).$$

Then, because $(T_2 \circ T_1)$ is one-to-one, it follows that $\mathbf{u} = \mathbf{v}$. Therefore T_1 is one-to-one.

Solution: Alternate Proof. Suppose that $T_2 \circ T_1 : U \to W$ is one-to-one. Then $\ker(T_2 \circ T_1) = \{\mathbf{0}\}$. Let $\mathbf{u} \in \ker(T_1)$. Then $T_1(\mathbf{u}) = \mathbf{0}$, and so

$$\begin{aligned} (T_2 \circ T_1)(\mathbf{u}) &= T_2(T_1(\mathbf{u})) \\ &= T_2(\mathbf{0}) && \text{(def'n of } \mathbf{u}\text{)} \\ &= \mathbf{0}. && \text{(property of linear transformations)} \end{aligned}$$

But then $\mathbf{u} \in \ker(T_2 \circ T_1)$, and so $\mathbf{u} = \mathbf{0}$. Thus since $\mathbf{u} \in \ker(T_1)$ we have that $\ker(T_1) = \{\mathbf{0}\}$. Thus T_1 is 1:1 as well.

**32. Let $T : V \to W$ be an onto linear transformation. Prove that if $\dim(V) = \dim(W)$, then T is an isomorphism.

Solution: Assume that $\dim(V) = \dim(W)$. Then since T is onto, $T(V) = W$, and so $\text{rank}(T) = \dim(W) = \dim(V)$. But then by the dimension theorem,

$$\begin{aligned} \text{nullity}(T) &= \dim(V) - \text{rank}(T) \\ &= \dim(V) - \dim(V) \\ &= 0. \end{aligned}$$

Therefore $\text{Ker}(T) = \{\mathbf{0}\}$, and therefore by the theorem above, we have that T is 1:1. Therefore T is both 1:1 and onto, and is thus an isomorphism.

**33. Let $T : V \to W$ be a one-to-one linear transformation. Prove that T is an isomorphism between V and $T(V)$.

Solution: T is given to be 1:1. Viewing it as a linear transformation between V and $T(V)$, it is also certainly onto by the definition of $T(V)$. Therefore it is 1:1 and onto, and is thus an isomorphism.

**34. Let E be a fixed 2×2 elementary matrix.

(a) Does the formula $T(A) = EA$ define a one-to-one linear operator on $M_{2,2}$? Prove or disprove.

(b) Does the formula $T(A) = EA$ define an onto linear operator on $M_{2,2}$? Prove or disprove.

Solution: 1-to-1: Let $A, B \in M_{2,2}$ be such that $T(A) = T(B)$. Then $EA = EB$. Since E is an elementary matrix, and all elementary matrices are invertible, E^{-1} exists. Multiplying both sides by E^{-1} we get $E^{-1}EA = E^{-1}EB$, and thus $A = B$. Therefore T is 1:1.

Onto: Let $A \in M_{2,2}$. Then $B = E^{-1}A$ is a matrix in $M_{2,2}$ such that $T(B) = EB = A$. Thus T is onto.

Thus T is 1:1 and onto (and is thus an isomorphism).

**35. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T : V \to W$ be a linear transformation. Show that if $T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_n) = \mathbf{0}_W$, then T is the zero transformation (that is, for every $\mathbf{v} \in V$, $T(\mathbf{v}) = \mathbf{0}_W$).

Solution: Let $\mathbf{v} \in V$. Then since B is a basis for V, there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$.

Then,

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T(c_1\mathbf{v}_1) + \ldots + T(c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) \\ &= c_1\mathbf{0}_W + \ldots + c_n\mathbf{0}_W \\ &= \mathbf{0}_W. \end{aligned}$$

Thus T is the zero transformation.

***36. Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Prove that if for all i, $1 \leq i \leq n$, $T_1(\mathbf{v}_i) = T_2(\mathbf{v}_i)$, then $T_1 = T_2$ (that is, for all $v \in V$, $T_1(v) = T_2(v)$).

Solution: Let $\mathbf{v} \in V$. Then since B is a basis for V, there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$.

Then,

$$\begin{aligned} T_1(\mathbf{v}) &= T_1(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T_1(c_1\mathbf{v}_1) + \ldots + T_1(c_n\mathbf{v}_n) \\ &= c_1T_1(\mathbf{v}_1) + \ldots + c_nT_1(\mathbf{v}_n) \\ &= c_1T_2(\mathbf{v}_1) + \ldots + c_nT_2(\mathbf{v}_n) \\ &= T_2(c_1\mathbf{v}_1) + \ldots + T_2(c_n\mathbf{v}_n) \\ &= T_2(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T_2(\mathbf{v}). \end{aligned}$$

Thus $T_1 = T_2$.

EIGENVALUE/VECTOR AND INNER PRODUCT SPACE PROOFS

37. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Let V be the set of all eigenvectors corresponding to λ, together with the zero vector. Prove that V is a subspace of $\mathbb{R}^n$.

Solution: Let E be the set of all eigenvectors corresponding to λ, together with the zero vector. Let $\mathbf{u}, \mathbf{v} \in E$, $k \in \mathbb{R}$.

- Closure under addition:

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= A\mathbf{u} + A\mathbf{v} \\ &= \lambda\mathbf{u} + \lambda\mathbf{v} \\ &= \lambda(\mathbf{u}+\mathbf{v}). \end{aligned}$$

  Therefore $\mathbf{u}+\mathbf{v}$ is also in E.

- Closure under scalar mult:

$$\begin{aligned} A(k\mathbf{u}) &= kA(\mathbf{u}) \\ &= k(\lambda\mathbf{u}) \\ &= \lambda(k\mathbf{u}). \end{aligned}$$

  Thus $k\mathbf{u} \in E$ as well.

Therefore by the subspace theorem, E is a subspace of $\mathbb{R}^n$.

38. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

39. Show that for all $\mathbf{u}, \mathbf{v}$ in an inner product space V, and $k \in \mathbb{R}$,

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$$

40. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}-\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$$

41. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}, \mathbf{v}-\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$$

42. Show that in any inner product space V, for all $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.

Solution: Let $\mathbf{v} \in V$. Then

$$\begin{aligned} \langle \mathbf{v}, \mathbf{0} \rangle &= \langle \mathbf{v}, 0 \cdot \mathbf{0} \rangle \\ &= 0 \langle \mathbf{v}, \mathbf{0} \rangle \\ &= 0. \end{aligned}$$

43. Prove each of the following properties about inner product spaces: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V, and all $k \in \mathbb{R}$,
    - $||\mathbf{u}|| \geq 0$
    - $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
    - $||k\mathbf{u}|| = |k|||\mathbf{u}||$
    - $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (Triangle Inequality)
    - $d(\mathbf{u}, \mathbf{v}) \geq 0$
    - $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
    - $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
    - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$. (Triangle Inequality)
44. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then so are $\frac{1}{||\mathbf{u}||}\mathbf{u}$ and $\frac{1}{||\mathbf{v}||}\mathbf{v}$.

*45. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.

Solution:

$$\begin{aligned} |\lambda I - A| &= |(\lambda I - A)^T| \\ &= |(\lambda I)^T - A^T| \\ &= |\lambda I - A^T|. \end{aligned}$$

Thus A and A^T have the same characteristic polynomials, and so must have the same eigenvalues.

**46. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

Solution: I will solve this problem by proving the contrapositives: that A is not invertible if and only if 0 is an eigenvalue of A.

($\Rightarrow$): Assume A is not invertible. Then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and thus at least one non-zero solution, say $\mathbf{x}_0$. Then

$$A\mathbf{x}_0 = 0 = 0\mathbf{x}_0$$

and thus 0 is an eigenvalue of A with eigenvector $\mathbf{x}_0$.

($\Leftarrow$): Assume 0 is an eigenvalue of A. Then $\det(A - 0I) = \det(A) = 0$. Thus A is not invertible.

**47. Prove that if $B = C^{-1}AC$, then B and A have the same eigenvalues (HINT: Look at the characteristic polynomials of B and A).

Solution:

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - C^{-1}AC| \\
&= |\lambda C^{-1}C - C^{-1}AC| \\
&= |C^{-1}(\lambda C - AC)| \\
&= |C^{-1}(\lambda I - A)C| \\
&= |C^{-1}|\ |\lambda I - A|\ |C| \\
&= |\lambda I - A|\ |C^{-1}|\ |C| \\
&= |\lambda I - A|\ |C^{-1}C| \\
&= |\lambda I - A|\ |I| \\
&= |\lambda I - A|.
\end{aligned}$$

Thus A and B have the same characteristic polynomials, and so must have the same eigenvalues.

**48. Let $\mathbf{v}$ be a nonzero vector in an inner product space V. Let W be the set of all vectors in V that are orthogonal to $\mathbf{v}$. Prove that W is a subspace of V.

Solution: Let $W = \{\mathbf{w} \in V : \langle \mathbf{w}, \mathbf{v} \rangle = 0\}$. Certainly W is non-empty since the zero vector is orthogonal to every vector in V.

Let $\mathbf{a}, \mathbf{b} \in W$, $k \in \mathbb{R}$.

A1. We need to check if $\mathbf{a} + \mathbf{b}$ is orthogonal to $\mathbf{v}$.

$$\begin{aligned}
\langle \mathbf{a} + \mathbf{b}, \mathbf{v} \rangle &= \langle \mathbf{a}, \mathbf{v} \rangle + \langle \mathbf{b}, \mathbf{v} \rangle \\
&= 0 + 0 = 0.
\end{aligned}$$

Thus $\mathbf{a} + \mathbf{b} \in W$.

M1. We need to check if $k\mathbf{a} \in W$.

$$\begin{aligned}
\langle k\mathbf{a}, \mathbf{v} \rangle &= k\langle \mathbf{a}, \mathbf{v} \rangle \\
&= k(0) = 0.
\end{aligned}$$

Thus $k\mathbf{a} \in W$.

Therefore by the subspace theorem, W is a subspace of V.

**49. Prove that for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space, if

$$||\mathbf{u}|| = ||\mathbf{v}||,$$

then $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$.

Solution: Assume that $||\mathbf{u}|| = ||\mathbf{v}||$. Then,

$$\begin{aligned}\langle \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle &= \langle \mathbf{u}, \mathbf{u}-\mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}-\mathbf{v}\rangle \\ &= \langle \mathbf{u}-\mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}-\mathbf{v}, \mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{v}\rangle \\ &= ||\mathbf{u}||^2 - \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - ||\mathbf{v}||^2 \\ &= ||\mathbf{u}||^2 - ||\mathbf{v}||^2 \\ &= ||\mathbf{u}||^2 - ||\mathbf{u}||^2 \\ &= 0.\end{aligned}$$

Thus $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{u}-\mathbf{v}$.

**50. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a basis for an inner product space V. Show that the zero vector is the only vector in V that is orthogonal to all of the basis vectors.

Solution: We have a property (proved elsewhere) that for all $\mathbf{v} \in V$ (and thus for all $\mathbf{v} \in B$), $\langle \mathbf{0}, \mathbf{v}\rangle = 0$. But this question is in some sense the opposite of this. Let $\mathbf{w} \in V$ be a vector such that for all $\mathbf{v} \in B$, $\langle \mathbf{w}, \mathbf{v}\rangle = 0$. Then we must prove that $\mathbf{w}$ must have been the zero vector.

Since B is a basis for V, we know that there exist $k_1, \ldots, k_r \in \mathbb{R}$ such that $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$.

Then look at $\langle \mathbf{w}, \mathbf{w}\rangle$:

$$\begin{aligned}\langle \mathbf{w}, \mathbf{w}\rangle &= \langle \mathbf{w}, k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r\rangle \\ &= \langle \mathbf{w}, k_1\mathbf{v}_1\rangle + \langle \mathbf{w}, k_2\mathbf{v}_2\rangle + \cdots + \langle \mathbf{w}, k_r\mathbf{v}_r\rangle \\ &= k_1\langle \mathbf{w}, \mathbf{v}_1\rangle + k_2\langle \mathbf{w}, \mathbf{v}_2\rangle + \cdots + k_r\langle \mathbf{w}, \mathbf{v}_r\rangle \\ &= k_1(0) + k_2(0) + \cdots + k_r(0) \\ &= 0.\end{aligned}$$

Thus since $\langle \mathbf{w}, \mathbf{w}\rangle = 0$, by positivity, we know that $\mathbf{w} = \mathbf{0}$.

***51. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V. Prove that

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n\rangle\mathbf{v}_n.$$

Solution: Let $\mathbf{u} \in V$. Since S is a basis, there exist $k_1, \ldots, k_n$ such that $\mathbf{u} = k_1\mathbf{v}_1 + \cdots + k_n\mathbf{v}_n$. Then

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle k_1\mathbf{v}_1 + \cdots + k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= \langle k_1\mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + \langle k_n\mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + k_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle \\
&= k_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle && \text{(since } S \text{ is orthogonal)} \\
&= k_i||\mathbf{v}_i||^2 \\
&= k_i && \text{(since } S \text{ is orthonormal).}
\end{aligned}$$

Thus $\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$.

***52. An $n \times n$ matrix A is said to be **nilpotent** if for some $k \in \mathbb{Z}^+$, A^k is a zero matrix. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

Solution: Let A be a nilpotent matrix and let $k \in \mathbb{Z}^+$ be such that A^k is a zero matrix. Let λ be a eigenvalue of A with eigenvector $\mathbf{x}$. Then

$$\begin{aligned}
A\mathbf{x} &= \lambda\mathbf{x} \\
A^2\mathbf{x} &= A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda^2\mathbf{x}. \\
&\vdots\vdots \\
A^k\mathbf{x} &= \lambda^k\mathbf{x}.
\end{aligned}$$

But A^k is a zero matrix, and so the left hand side is a zero matrix. Thus $\lambda^k\mathbf{x}$ is a zero matrix. However $\mathbf{x}$ being an eigenvector forces $\mathbf{x} \neq \mathbf{0}$, and thus $\lambda^k = 0$, and so $\lambda = 0$. Thus 0 is the only eigenvalue of A.

***53. Let W be any subspace of an inner product space V, $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ an orthonormal basis for W. Let $\mathbf{v} \in V$. Let the vector $\mathbf{v}_0$ be defined as

$$\mathbf{v}_0 = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \cdots + \langle \mathbf{v}, \mathbf{b}_n \rangle \mathbf{b}_n = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \mathbf{b}_i.$$

Certainly $\mathbf{v}_0 \in W$. Prove that $\mathbf{v} - \mathbf{v}_0$ is orthogonal to every vector in W.

Solution: What we need to check is the inner product of $\mathbf{v} - \mathbf{v}_0$ with every vector in W and verify it is 0.

Let $\mathbf{w} \in W$. Then since B is an orthonormal basis, $\mathbf{w} = \sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{b}_i \rangle \mathbf{b}_i$. Then

$$\langle \mathbf{v} - \mathbf{v}_0, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle - \langle \mathbf{v}_0, \mathbf{w} \rangle,$$

and

$$\begin{aligned}
\langle \mathbf{v}_0, \mathbf{w}\rangle &= \langle \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \mathbf{b}_i, \mathbf{w}\rangle && \text{(by def'n of } \mathbf{v}_0\text{)}\\
&= \sum_{i=1}^{n} \langle \langle \mathbf{v}, \mathbf{b}_i\rangle \mathbf{b}_i, \mathbf{w}\rangle && \text{(by additivity)}\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \langle \mathbf{b}_i, \mathbf{w}\rangle && \text{(by homogeneity)}\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \langle \mathbf{b}_i, \sum_{j=1}^{n} \langle \mathbf{w}, \mathbf{b}_j\rangle \mathbf{b}_j\rangle\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \sum_{j=1}^{n} \langle \mathbf{b}_i, \langle \mathbf{w}, \mathbf{b}_j\rangle \mathbf{b}_j\rangle && \text{(by additivity)}\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \sum_{j=1}^{n} \langle \mathbf{w}, \mathbf{b}_j\rangle \langle \mathbf{b}_i, \mathbf{b}_j\rangle && \text{(by homogeneity)}\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \langle \mathbf{w}, \mathbf{b}_i\rangle \langle \mathbf{b}_i, \mathbf{b}_i\rangle && \text{(since } i \neq j \implies \langle \mathbf{b}_i, \mathbf{b}_j\rangle = 0\text{)}\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \langle \mathbf{w}, \mathbf{b}_i\rangle ||\mathbf{b}_i||^2\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i\rangle \langle \mathbf{w}, \mathbf{b}_i\rangle && \text{(since } \mathbf{b}_i \text{ is a unit vector)}\\
&= \sum_{i=1}^{n} \langle \mathbf{v}, \langle \mathbf{w}, \mathbf{b}_i\rangle \mathbf{b}_i\rangle && \text{(by homogeneity)}\\
&= \langle \mathbf{v}, \sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{b}_i\rangle \mathbf{b}_i\rangle && \text{(by additivity)}\\
&= \langle \mathbf{v}, \mathbf{w}\rangle.
\end{aligned}$$

Therefore, for any $\mathbf{w} \in W$,

$$\begin{aligned}
\langle \mathbf{v} - \mathbf{v}_0, \mathbf{w}\rangle &= \langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{v}_0, \mathbf{w}\rangle\\
&= \langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{w}\rangle\\
&= 0.
\end{aligned}$$

Thus $\mathbf{v} - \mathbf{v}_0$ is orthogonal to everything in W.