MATH 2300 Sample Proofs

This document contains a number of theorems, the proofs of which are at a difficulty level where they could be put on a test or exam. This should not be taken as an indication that the only theorems on tests or exams will be taken from this document, nor that every (or any) theorem in this document need be tested. This document is for information purposes only. Questions marked with a * are a little harder than the others, and the more stars, the harder the question, but **all are testable**. More solutions will be provided as time allows.

VECTOR SPACE PROOFS

1. Prove that for any set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in a vector space V, $\text{span}(S)$ is a subspace of V.

2. Prove that if S is a linearly independent set of vectors, then S is a basis for $\text{span}(S)$.

3. Show that if A is an $m \times n$ matrix, then the solution set V to the equation $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

4. Prove that any finite set of vectors containing the zero vector is linearly dependent.

5. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v} \in V$ can be expressed as a linear combination of the elements of S in exactly one way.

*6. Given that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of vectors in some vector space V, prove that:
   (a) the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.
   (b) the set $\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$ is linearly independent.
   (c) the set $\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}\}$ is linearly independent.

*7. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ be such that $\mathbf{u} \bullet \mathbf{v} = 0$. Prove that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set.

*8. Let V be a vector space. Prove that for every $\mathbf{u} \in V$, $0 \cdot \mathbf{u} = \mathbf{0}$.

*9. Let V be a vector space. Prove that for every $k \in \mathbb{R}$, $k \cdot \mathbf{0} = \mathbf{0}$.

*10. Let V be a vector space. Prove that for every $\mathbf{u} \in V$, $(-1) \cdot \mathbf{u} = -\mathbf{u}$.

*11. Let V be a vector space. Prove that if for some $k \in \mathbb{R}$ and $\mathbf{u} \in V$, $k \cdot \mathbf{u} = 0$, then either $k = 0$, or $\mathbf{u} = 0$.

*12. Prove that a set of vectors is linearly dependent if and only if at least one vector in the set is a linear combination of the others.

*13. Let A be a $m \times n$ matrix. Prove that if both the set of rows of A and the set of columns of A form linearly independent sets, then A must be square.

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**14. Let V be the set of 2×2 matrices, together with the operation $\oplus$ defined for any 2×2 matrices A and B as

$$A \oplus B = AB \text{ (the usual matrix multiplication)},$$

and with the standard scalar multiplication for matrices.

(a) Show that the vector space axiom $A4$ holds.

(b) Prove that V is **not** a vector space.

**15. Let

$$V = \{(a, b) \in \mathbb{R}^2 : a > 0, b > 0\}$$

together with the operations defined as follows: for $(a, b), (c, d) \in V$, $k \in \mathbb{R}$,

$$(a, b) \oplus (c, d) = (ac, bd)$$

$$k \cdot (a, b) = (a^k, b^k).$$

(a) Show that the vector space axiom $M3$ holds in this space.

(b) Does the axiom $A4$ hold in this space? If so, find the zero vector and prove it is the zero vector. If not, show that there is no possible zero vector.

**16. Let V be a vector space, and let W_1 and W_2 be subspaces of V. Prove that the set

$$U = \{\mathbf{v} : \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}$$

(that is, U is the set of vectors in BOTH W_1 and W_2). Prove that U is a subspace of V as well.

**17. Let W be a subspace of a vector space V, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in W$. Prove then that every linear combination of these vectors is also in W.

**18. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

LINEAR TRANSFORMATION PROOFS

19. Prove that the range of a linear transformation $T : V \to W$ is a subspace of W.

20. Prove that given two linear transformations $T_1 : U \to V$ and $T_2 : V \to W$, the composition $T_2 \circ T_1 : U \to W$ is also a linear transformation.

21. Prove that for any linear transformation $T : V \to W$, $\ker(T)$ is a subspace of W.

22. Prove that If $T_1 : U \to V$ is one-to-one, and $T_2 : V \to W$ is one-to-one, then the composition $T_2 \circ T_1 : U \to W$ is also one-to-one.

23. If $T_1 : U \to V$ is onto, and $T_2 : V \to W$ is onto, then the composition $T_2 \circ T_1 : U \to W$ is also onto.

24. Prove that for any one-to-one linear transformation $T : V \to W$, T^{-1} is also a one-to-one linear transformation.

25. Prove that for any $m \times n$ matrix M, $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation.

*26. If $T : V \to W$ is a linear transformation, then prove each of the following:

- If T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$.
- If $\ker(T) = \{\mathbf{0}\}$, then T is one-to-one.

*27. If V is a finite-dimensional vector space, and $T : V \to V$ is a linear operator, then prove that if T is one-to-one, then the range of T is all of V.

*28. Prove that if $T : V \to W$ is an isomorphism between V and W (one-to-one and onto), and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then $T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}$ is a basis for W.

*29. Prove that every vector space of dimension n is isomorphic to $\mathbb{R}^n$.

**30. Let $T : V \to W$ be a one-to-one linear transformation. Prove that if $\dim(V) = \dim(W)$ (and both V and W are finite-dimensional), then T is an isomorphism.

**31. Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations. Prove that if $T_2 \circ T_1$ is one-to-one, then T_1 must be one-to-one.

**32. Let $T : V \to W$ be an onto linear transformation. Prove that if $\dim(V) = \dim(W)$, then T is an isomorphism.

**33. Let $T : V \to W$ be a one-to-one linear transformation. Prove that T is an isomorphism between V and $T(V)$.

**34. Let E be a fixed 2×2 elementary matrix.

(a) Does the formula $T(A) = EA$ define a one-to-one linear operator on $M_{2,2}$? Prove or disprove.

(b) Does the formula $T(A) = EA$ define an onto linear operator on $M_{2,2}$? Prove or disprove.

**35. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T : V \to W$ be a linear transformation. Show that if $T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_n) = \mathbf{0}_W$, then T is the zero transformation (that is, for every $\mathbf{v} \in V$, $T(\mathbf{v}) = \mathbf{0}_W$).

***36. Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations and let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Prove that if for all i, $1 \leq i \leq n$, $T_1(\mathbf{v}_i) = T_2(\mathbf{v}_i)$, then $T_1 = T_2$ (that is, for all $v \in V$, $T_1(v) = T_2(v)$).

EIGENVALUE/VECTOR AND INNER PRODUCT SPACE PROOFS

37. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Let V be the set of all eigenvectors corresponding to λ, together with the zero vector. Prove that V is a subspace of $\mathbb{R}^n$.

38. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$$

39. Show that for all $\mathbf{u}, \mathbf{v}$ in an inner product space V, and $k \in \mathbb{R}$,

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$$

40. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$$

41. Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V,

$$\langle \mathbf{u}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{w} \rangle$$

42. Show that in any inner product space V, for all $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.

43. Prove each of the following properties about inner product spaces: for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an inner product space V, and all $k \in \mathbb{R}$,

    - $||\mathbf{u}|| \geq 0$
    - $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
    - $||k\mathbf{u}|| = |k|||\mathbf{u}||$
    - $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (Triangle Inequality)
    - $d(\mathbf{u}, \mathbf{v}) \geq 0$
    - $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
    - $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
    - $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$. (Triangle Inequality)

44. Prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then so are $\frac{1}{||\mathbf{u}||}\mathbf{u}$ and $\frac{1}{||\mathbf{v}||}\mathbf{v}$.

*45. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.

**46. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

**47. Prove that if $B = C^{-1}AC$, then B and A have the same eigenvalues (HINT: Look at the characteristic polynomials of B and A).

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**48. Let $\mathbf{v}$ be a nonzero vector in an inner product space V. Let W be the set of all vectors in V that are orthogonal to $\mathbf{v}$. Prove that W is a subspace of V.

**49. Prove that for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space, if

$$||\mathbf{u}|| = ||\mathbf{v}||,$$

then $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{u} - \mathbf{v}$.

**50. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a basis for an inner product space V. Show that the zero vector is the only vector in V that is orthogonal to all of the basis vectors.

***51. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space V, and $\mathbf{u}$ is any vector in V. Prove that

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

***52. An $n \times n$ matrix A is said to be **nilpotent** if for some $k \in \mathbb{Z}^+$, A^k is a zero matrix. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

***53. Let W be any subspace of an inner product space V, $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ an orthonormal basis for W. Let $\mathbf{v} \in V$. Let the vector $\mathbf{v}_0$ be defined as

$$\mathbf{v}_0 = \langle \mathbf{v}, \mathbf{b}_1 \rangle \mathbf{b}_1 + \cdots + \langle \mathbf{v}, \mathbf{b}_n \rangle \mathbf{b}_n = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{b}_i \rangle \mathbf{b}_i.$$

Certainly $\mathbf{v}_0 \in W$. Prove that $\mathbf{v} - \mathbf{v}_0$ is orthogonal to every vector in W.