MATH 2300 Assignment 5 10R

Attempt all questions and show all your work. Due April 9, 2010. Hand in your paper in the box outside my office door (425 Machray Hall) **BEFORE 1:30pm**.

1. For each of the following matrices, find the characteristic polynomial, the eigenvalues, and the corresponding eigenspaces for each eigenvalue:

   (a) $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

   Solution: Characteristic polynomial: $x^2 - 6x + 9$
   Eigenvalues: 3
   Basis for eigenspace of eigenvalue 3: $\{(1, 1)\}$

   (b) $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}$

   Solution: Characteristic polynomial: $x^3 - 5x^2 + 7x - 3$
   Real eigenvalues: $1, 3$
   Basis for eigenspace of eigenvalue 1: $\{(1, 1, 1)\}$
   Basis for eigenspace of eigenvalue 3: $\{(0, 1, 1)\}$

   (c) $\begin{bmatrix} 4 & 2 & -2 & 2 \\ 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -3 & 5 \end{bmatrix}$ 1-

   Solution: Characteristic polynomial: $x^4 - 14x^3 + 68x^2 - 136x + 96$
   Real eigenvalues: $2, 4, 6$
   Basis for eigenspace of eigenvalue 2: $\{(-1, 1, 0, 0), (0, 0, 1, 1)\}$
   Basis for eigenspace of eigenvalue 4: $\{(0, -1, 0, 1)\}$
   Basis for eigenspace of eigenvalue 6: $\{(1, 0, 0, 1)\}$

2. What are the eigenvalues and eigenvectors of I_n?

   Solution: $|\lambda I_n - I_n| = |(\lambda - 1)I_n| = (\lambda - 1)^n$, which is equal to zero if and only if $\lambda = 1$. Therefore the only eigenvalue is $\lambda = 1$.

   However, when $\lambda = 1$, $\lambda I_n - I_n = \mathbf{0}$. This is the coefficient matrix for a system of equations. However, every vector is a solution to $\mathbf{0x} = \mathbf{0}$. Therefore a basis for the eigenspace of eigenvalue 1 is the standard basis for $\mathbb{R}^n$:

   $$\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}.$$

3. Compute A^9 using the process of diagonalization if $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$.

Solution: Characteristic polynomial: $x^2 - x - 2$

Real eigenvalues: $-1, 2$.

Basis for eigenspace of eigenvalue -1: $(-2, 1)$
Basis for eigenspace of eigenvalue 2: $(-1, 1)$

Therefore $P = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ diagonalizes $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$, and $P^{-1}AP = D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.

Finding P^{-1}:

$$\left[\begin{array}{cc|cc} -2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$
$$R_1 \leftarrow R_1 + 3R_2$$
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 1 & 1 & 0 & 1 \end{array}\right]$$
$$R_2 \leftarrow R_2 - R_1$$
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \end{array}\right]$$
$$R_2 \leftarrow -R_2$$
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right]$$
$$R_1 \leftarrow R_1 - 2R_2$$
$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

Therefore $P^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$.

Then $A = PDP^{-1}$, and $A^9 = PD^9P^{-1}$.

Thus

$$\begin{aligned} A^9 &= PD^9P^{-1} \\ &= P \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^9 P^{-1} \\ &= P \begin{bmatrix} (-1)^9 & 0 \\ 0 & 2^9 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} -1 & 0 \\ 0 & 512 \end{bmatrix} P^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 512 \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} 2 & -512 \\ -1 & 512 \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} 2 & -512 \\ -1 & 512 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} -514 & -1026 \\ 513 & 1025 \end{bmatrix}.
\end{aligned}
$$

4. Show that $\begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$ is not diagonalizable.

Solution: Characteristic polynomial: $x^2 - 4x + 4$

Real eigenvalues: 2

Basis for eigenspace of eigenvalue 2: $\{(1, 1)\}$

Since it is impossible to choose 2 linearly independent eigenvectors of $\begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$, we have that it is not diagonalizable.

5. Consider the vector space P_n of polynomials of degree less than or equal to n. Let f and g be two elements of P_n.

   (a) Prove that

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\ dx$$

   defines an inner product on P_n (note this is defined on P_n where n is arbitrary, and thus you must prove this in general).

Solution:

- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

  Let $f, g \in P_n$. Then

$$
\begin{aligned}
\langle f, g \rangle &= \int_0^1 f(x)g(x)\ dx \\
&= \int_0^1 g(x)f(x)\ dx \\
&= \langle g, f \rangle.
\end{aligned}
$$

- $\langle \mathbf{u}+\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$

  Let $f, g, h \in P_n$. Then

$$\begin{aligned}
\langle f+g, h\rangle &= \int_0^1 (f+g)(x)h(x)\ dx \\
&= \int_0^1 (f(x)+g(x))h(x)\ dx \\
&= \int_0^1 f(x)h(x) + g(x)h(x)\ dx \\
&= \int_0^1 f(x)h(x)\ dx + \int_0^1 g(x)h(x)\ dx \\
&= \langle f, h\rangle + \langle g, h\rangle.
\end{aligned}$$

- $\langle k\mathbf{u}, \mathbf{v}\rangle = k\langle \mathbf{u}, \mathbf{v}\rangle$

  Let $f, g \in P_n$, $k \in \mathbb{R}$. Then

$$\begin{aligned}
\langle kf, g\rangle &= \int_0^1 (kf)(x)g(x)\ dx \\
&= \int_0^1 k(f(x))g(x)\ dx \\
&= k\int_0^1 f(x)g(x)\ dx \\
&= k\langle f, g\rangle.
\end{aligned}$$

- $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}_V$.

  Let $f \in P_n$. Then

$$\begin{aligned}
\langle f, f\rangle &= \int_0^1 f(x)f(x)\ dx \\
&= \int_0^1 f^2(x)\ dx.
\end{aligned}$$

  Since this is the net area between the curve $f^2(x)$ and the x-axis, and since $f^2(x)$ is always positive, we know that this value is always non-negative.

  Further, if this area is zero, then since $f^2(x)$ is always non-negative, it must be that $f^2(x)$, and thus $f(x)$, is the zero function, and thus $f(x) = 0 = \mathbf{0}$.

(b) Find $\langle x^2+2x-1, 4x+1\rangle$

Solution:

$$\begin{aligned}
\langle x^2+2x-1, 4x+1\rangle &= \int_0^1 (x^2+2x-1)(4x+1)\ dx \\
&= \int_0^1 4x^3+x^2+8x^2+2x-4x-1\ dx \\
&= \int_0^1 4x^3+9x^2-2x-1\ dx \\
&= (x^4+3x^3-x^2-x)\Big|_0^1 \\
&= (1^4+3(1)^3-(1)^2-1)-(0^4+3(0)^3-(0)^2-0) \quad = 1+3-1-1 \\
&= 2.
\end{aligned}$$

(c) Show that $f(x)=1$ and $g(x)=\frac{1}{2}-x$ are orthogonal

Solution:

$$\begin{aligned}
\langle 1, \frac{1}{2}-x\rangle &= \int_0^1 (1)(\frac{1}{2}-x)\ dx \\
&= \int_0^1 \frac{1}{2}-x\ dx \\
&= (\frac{1}{2}x-\frac{1}{2}x^2)\Big|_0^1 \\
&= (\frac{1}{2}-\frac{1}{2})-0 \\
&= 0.
\end{aligned}$$

Thus f and g are orthogonal.

(d) Determine which of the functions $g(x)=x^2+2x-3$ or $h(x)=x^2-3x+4$ is closest to the function $f(x)=x^2$.

Solution:

$$\begin{aligned}
d(f,g) &= ||f-g|| \\
&= ||x^2-(x^2+2x-3)|| \\
&= ||-2x+3|| \\
&= \sqrt{\langle -2x+3, -2x+3\rangle} \\
&= \sqrt{\int_0^1 (-2x+3)(-2x+3)\ dx} \\
&= \sqrt{\int_0^1 4x^2-12x+9\ dx}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\left(\frac{4}{3}x^3 - 6x^2 + 9x\right)\Big|_0^1} \\
&= \sqrt{\frac{4}{3} - 6 + 9} \\
&= \sqrt{\frac{4}{3} + 3} \\
&= \sqrt{\frac{13}{3}}.
\end{aligned}
$$

$$
\begin{aligned}
d(f,h) &= ||f - h|| \\
&= ||x^2 - (x^2 - 3x + 4)|| \\
&= ||3x - 4|| \\
&= \sqrt{\langle 3x - 4, 3x - 4\rangle} \\
&= \sqrt{\int_0^1 (3x-4)(3x-4)\ dx} \\
&= \sqrt{\int_0^1 9x^2 - 24x + 16\ dx} \\
&= \sqrt{(3x^3 - 12x^2 + 16x)|_0^1} \\
&= \sqrt{3 - 12 + 16} \\
&= \sqrt{7} > \sqrt{\frac{13}{3}}.
\end{aligned}
$$

Therefore f is closer to g than to h.

6. Show that in any inner product space V, for all $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.

Solution: Let $\mathbf{v} \in V$. Then

$$
\begin{aligned}
\langle \mathbf{v}, \mathbf{0} \rangle &= \langle \mathbf{v}, 0 \cdot \mathbf{0} \rangle \\
&= 0\langle \mathbf{v}, \mathbf{0} \rangle \\
&= 0.
\end{aligned}
$$

7. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Let V be the set of all eigenvectors corresponding to λ, together with the zero vector. Prove that V is a subspace of $\mathbb{R}^n$.

Solution: Let E be the set of all eigenvectors corresponding to λ, together with the zero vector. Let $\mathbf{u}, \mathbf{v} \in E$, $k \in \mathbb{R}$.

- Closure under addition:

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= A\mathbf{u} + A\mathbf{v} \\ &= \lambda\mathbf{u} + \lambda\mathbf{v} \\ &= \lambda(\mathbf{u}+\mathbf{v}). \end{aligned}$$

  Therefore $\mathbf{u}+\mathbf{v}$ is also in E.

- Closure under scalar mult:

$$\begin{aligned} A(k\mathbf{u}) &= kA(\mathbf{u}) \\ &= k(\lambda\mathbf{u}) \\ &= \lambda(k\mathbf{u}). \end{aligned}$$

  Thus $k\mathbf{u} \in E$ as well.

Therefore by the subspace theorem, E is a subspace of $\mathbb{R}^n$.

8. Let A be an $n \times n$ matrix. Prove that A and A^T have the same eigenvalues.

Solution:

$$\begin{aligned} |\lambda I - A| &= |(\lambda I - A)^T| \\ &= |(\lambda I)^T - A^T| \\ &= |\lambda I - A^T|. \end{aligned}$$

Thus A and A^T have the same characteristic polynomials, and so must have the same eigenvalues.

9. Let A be an $n \times n$ matrix. Prove that A is invertible if and only if 0 is not an eigenvalue of A.

Solution: I will solve this problem by proving the contrapositives: that A is not invertible if and only if 0 is an eigenvalue of A.

($\Rightarrow$): Assume A is not invertible. Then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and thus at least one non-zero solution, say $\mathbf{x}_0$. Then

$$A\mathbf{x}_0 = 0 = 0\mathbf{x}_0$$

and thus 0 is an eigenvalue of A with eigenvector $\mathbf{x}_0$.

($\Leftarrow$): Assume 0 is an eigenvalue of A. Then $\det(A - 0I) = \det(A) = 0$. Thus A is not invertible.

10. An $n \times n$ matrix A is said to be **nilpotent** if for some $k \in \mathbb{Z}^+$, A^k is a zero matrix. Prove that if A is nilpotent, then 0 is the only eigenvalue of A.

Solution: Let A be a nilpotent matrix and let $k \in \mathbb{Z}^+$ be such that A^k is a zero matrix. Let λ be a eigenvalue of A with eigenvector $\mathbf{x}$. Then

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^2\mathbf{x} &= A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda^2\mathbf{x}. \\ &\vdots \\ A^k\mathbf{x} &= \lambda^k\mathbf{x}. \end{aligned}$$

But A^k is a zero matrix, and so the left hand side is a zero matrix. Thus $\lambda^k\mathbf{x}$ is a zero matrix. However $\mathbf{x}$ being an eigenvector forces $\mathbf{x} \neq \mathbf{0}$, and thus $\lambda^k = 0$, and so $\lambda = 0$. Thus 0 is the only eigenvalue of A.

11. Prove that if $B = C^{-1}AC$, then B and A have the same eigenvalues (HINT: Look at the characteristic polynomials of B and A).

Solution:

$$\begin{aligned} |\lambda I - B| &= |\lambda I - C^{-1}AC| \\ &= |\lambda C^{-1}C - C^{-1}AC| \\ &= |C^{-1}(\lambda C - AC)| \\ &= |C^{-1}(\lambda I - A)C| \\ &= |C^{-1}|\ |\lambda I - A|\ |C| \\ &= |\lambda I - A|\ |C^{-1}|\ |C| \\ &= |\lambda I - A|\ |C^{-1}C| \\ &= |\lambda I - A|\ |I| \\ &= |\lambda I - A|. \end{aligned}$$

Thus A and B have the same characteristic polynomials, and so must have the same eigenvalues.

12. Let $\mathbf{v}$ be a nonzero vector in an inner product space V. Let W be the set of all vectors in V that are orthogonal to $\mathbf{v}$. Prove that W is a subspace of V.

Solution: Let $W = \{\mathbf{w} \in V : \langle \mathbf{w}, \mathbf{v} \rangle = 0\}$. Certainly W is non-empty since the zero vector is orthogonal to every vector in V.

Let $\mathbf{a}, \mathbf{b} \in W$, $k \in \mathbb{R}$.

A1. We need to check if $\mathbf{a} + \mathbf{b}$ is orthogonal to $\mathbf{v}$.

$$\begin{aligned} \langle \mathbf{a} + \mathbf{b}, \mathbf{v} \rangle &= \langle \mathbf{a}, \mathbf{v} \rangle + \langle \mathbf{b}, \mathbf{v} \rangle \\ &= 0 + 0 = 0. \end{aligned}$$

Thus $\mathbf{a} + \mathbf{b} \in W$.

M1. We need to check if $k\mathbf{a} \in W$.

$$\begin{aligned} \langle k\mathbf{a}, \mathbf{v} \rangle &= k\langle \mathbf{a}, \mathbf{v} \rangle \\ &= k(0) = 0. \end{aligned}$$

Thus $k\mathbf{a} \in W$.

Therefore by the subspace theorem, W is a subspace of V.