UNIVERSITY OF MANITOBA

DATE: February 4, 2010

DEPARTMENT & COURSE NO: MATH 2300

EXAMINATION: Linear Algebra II

TERM TEST 1 SOLUTIONS

TITLE PAGE

TIME: 75 minutes

EXAMINER: Borgersen

NAME: (Print in ink) ____________________

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 75 minute exam. **Please show your work clearly.**

No texts or notes are permitted. No calculators are permitted. Cell phones, electronic translators, and other electronic devices are **not** permitted.

This exam has a title page and 10 pages of questions, including 2 blank pages for rough work and 1 page showing the axioms of a vector space. Please check that you have all the pages. You may remove the blank pages and axiom page if you want, but be careful not to loosen the staple.

The value of each question is indicated beside the statement of the question. The total value of all questions is 70 points.

If you need more scrap paper, use the back of the question pages.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	70	

True or False Questions

1. [20 points] Are the following true or false? (Write "True" or "False" on the line to the right). **These are marked right minus wrong, so if you don't know, don't guess**. Two marks each.

(a) If S is a subspace of a vector space V, and $\mathbf{0}$ is the zero vector in V, then it must be that $\mathbf{0}$ is also in S.

(a) TRUE

(b) There exists a subspace W of $\mathbb{R}^2$ that contains the vectors $(1,0)$ and $(0,1)$, and yet $W \neq \mathbb{R}^2$.

(b) FALSE

(c) For any matrix A, if the row vectors and the column vectors both form linearly independent sets then A must be square.

(c) TRUE

(d) The set V of 2×2 lower triangular matrices is a vector space of dimension 2.

(d) FALSE

(e) If two non-zero vectors in $\mathbb{R}^n$ are orthogonal, they must be linearly independent.

(e) TRUE

(f) The set $\{1, x^2, 1 + x^3\}$ forms a basis for P_3.

(f) FALSE

(g) The set of all vectors in $\mathbb{R}^2$ that are perpendicular to $(1,2)$ is a vector space using the usual vector addition and scalar multiplication.

(g) TRUE

(h) The only subspaces of $\mathbb{R}^3$ are lines and planes that contain the origin.

(h) FALSE

(i) The set $V = \{f(x) \in P_3 : f(1) = 1\}$ is a subspace of P_3.

(i) FALSE

(j) The set

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 2, \quad a, b, c, d \in \{0, 1\} \right\}$$

is linearly independent in $M_{2,2}$.

(j) FALSE

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2. [10 points] Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Prove that every vector $\mathbf{v} \in V$ can be expressed as a linear combination of the elements of S in exactly one way.

Solution: Let $\mathbf{v} \in V$. Let $c_1, \ldots, c_n, k_1, \ldots, k_n \in \mathbb{R}$ be such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

and

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n.$$

Then subtracting straight down we get

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \ldots + (c_n - k_n)\mathbf{v}_n.$$

But since S is a basis, S is linearly independent, and thus the only way this could happen is if $c_1 - k_1 = 0$, ..., $c_n - k_n = 0$. Thus $c_1 = k_1, \ldots, c_n = k_n$ and so $\mathbf{v}$ can only be written as a linear combination of the elements of S in one way.

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3. [10 points] Let V be a vector space, and let W_1 and W_2 be subspaces of V. Let

$$U = \{\mathbf{v} : \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}$$

(that is, U is the set of vectors in BOTH W_1 and W_2). Prove that U is a subspace of V as well. Explicitly refer to any and all axioms you use along the way.

Solution: A1) Let $\mathbf{u}, \mathbf{v} \in U$. Then $\mathbf{u}, \mathbf{v} \in W_1$ and $\mathbf{u}, \mathbf{v} \in W_2$. Then since $A1$ holds in W_1, $\mathbf{u} + \mathbf{v} \in W_1$, and since $A1$ holds in W_2, $\mathbf{u} + \mathbf{v} \in W_2$ as well. Thus $\mathbf{u} + \mathbf{v} \in U$ and so $A1$ holds.

M1) Let $\mathbf{u} \in U$, and let $k \in \mathbb{R}$. Then $\mathbf{u} \in W_1$ and $\mathbf{u} \in W_2$, and so since $M1$ holds in W_1, $k\mathbf{u} \in W_1$, and since $M1$ holds in W_2, $k\mathbf{u} \in W_2$ as well. Thus $k\mathbf{u} \in U$, and so $M1$ holds.

Thus, but the subspace theorem, U is a subspace of V.

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4. Let $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (3, -1, 0) \in \mathbb{R}^3$. Let $\mathbf{w} = (5, 3, 2)$ and $\mathbf{u} = (7, 0, 0)$.

   (a) [4 points] Show that $\mathbf{w} \in \mathrm{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$

Solution: It would suffice to just point out that

$$2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}.$$

If you can't "see" this, then you would have to set up a system of equations and solve it. Either way, it's pretty quick.

   (b) [6 points] Show that $\mathbf{u} \notin \mathrm{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$

Solution: Let $c_1, c_2 \in \mathbb{R}$ be such that $(7, 0, 0) = c_1(1, 2, 1) + c_2(3, -1, 0)$. Then

$$(7, 0, 0) = (c_1 + 3c_2, 2c_1 - c_2, c_1),$$

and so we have the system of equations:

$$\begin{aligned} c_1 + 3c_2 &= 7 \\ 2c_1 - c_2 &= 0 \\ c_1 &= 0. \end{aligned}$$

But since $c_1 = 0$ (from the third components), we then have that $3c_2 = 7$ and $-c_2 = 0$, which cannot both be true at the same time. Thus this system has no solution, and therefore $\mathbf{u} \notin \mathrm{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$.

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5. Let

$$S = \left\{ \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} : a, b \in \mathbb{R} \right\} \subseteq M_{2,2}.$$

(a) [4 points] Prove that S a subspace of $M_{2,2}$.

Solution: To show that S is a subspace of $M_{2,2}$, we must show that it is closed under addition and under scalar multiplication.

1). Let $\begin{bmatrix} a & b \\ a+b & b \end{bmatrix}, \begin{bmatrix} c & d \\ c+d & d \end{bmatrix} \in S$. Then

$$\begin{bmatrix} a & b \\ a+b & b \end{bmatrix} + \begin{bmatrix} c & d \\ c+d & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ (a+b)+(c+d) & b+d \end{bmatrix}$$
$$= \begin{bmatrix} a+c & b+d \\ (a+c)+(b+d) & b+d \end{bmatrix} \in S.$$

Thus S is closed under addition.

2). Let $\begin{bmatrix} a & b \\ a+b & b \end{bmatrix} \in S$, $k \in \mathbb{R}$. Then

$$k \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} = \begin{bmatrix} ka & kb \\ k(a+b) & kb \end{bmatrix} = \begin{bmatrix} ka & kb \\ ka+kb & kb \end{bmatrix} \in S.$$

Thus S is also closed under scalar multiplication. Thus by the subspace theorem, S is a subspace of $M_{2,2}$.

(b) [4 points] Find a basis for S, and prove your answer is a basis.

Solution:

$$\begin{aligned} S &= \left\{ \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & b \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \operatorname{span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\} \right). \end{aligned}$$

Thus the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

certainly spans S. It just remains to show that the set is linearly independent. But, since these two matrices are not scalar multiples of each other, they are linearly independent. Thus these two matrices form a basis.

(c) [2 points] What is the dimension of S?

Solution: The dimension of this space is 2, the number of elements in the found basis.

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6. [10 points] Show that
$$S = \{x, 2 + x^2, 5 + x\}$$
forms a basis for P_2.

Solution: Since P_2 has dimension 3 (take, say, the standard basis $\{1, x, x^2\}$), and since S has 3 elements, we know that either S is a basis or S is not linearly independent (that is, if S is linearly independent, then it MUST span the space since it has the same number of elements as the dimension of the space). So it suffices to show that S is linearly independent.

Let $c_1, c_2, c_3 \in \mathbb{R}$ be such that
$$c_1x + c_2(2 + x^2) + c_3(5 + x) = 0.$$

Then
$$(2c_2 + 5c_3) + (c_1 + c_3)x + (c_2)x^2 = 0 + 0x + 0x^2,$$

and we get the system of equations:

$$\begin{aligned} 2c_2 + 5c_3 &= 0 \\ c_1 + c_3 &= 0 \\ c_2 &= 0. \end{aligned}$$

Since $c_2 = 0$, we have from the first equation that $5c_3 = 0$ and so $c_3 = 0$. Then by the second equation we have that $c_1 = 0$ as well. Thus the only solution to this system is $c_1 = c_2 = c_3 = 0$, and so S is linearly independent.

Thus since S is a linearly independent set of 3 vectors in P_2 (a 3-D space), we know that S is a basis for P_2.

The Axioms of a Vector Space

A set V together with addition "$\oplus$" and scalar multiplication "$\cdot$" is a **Vector Space** if and only if each of the following axioms hold:

A1. for every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \oplus \mathbf{v} \in V$,

A2. for every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$,

A3. for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$,

A4. there exists an element $\mathbf{0} \in V$ such that for every $\mathbf{u} \in V$, $\mathbf{0} \oplus \mathbf{u} = \mathbf{u}$,

A5. for every $\mathbf{u} \in V$, there exists a "$-\mathbf{u}$" $\in V$ such that $\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}$,

M1. for every $\mathbf{u} \in V$ and $k \in \mathbb{R}$, $k \cdot \mathbf{u} \in V$,

M2. for every $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$, $k \cdot (\mathbf{u} \oplus \mathbf{v}) = k \cdot \mathbf{u} \oplus k \cdot \mathbf{v}$,

M3. for every $\mathbf{u} \in V$, and $k, m \in \mathbb{R}$, $(k+m) \cdot \mathbf{u} = k \cdot \mathbf{u} \oplus m \cdot \mathbf{u}$,

M4. for every $\mathbf{u} \in V$, and $k, m \in \mathbb{R}$, $k \cdot (m \cdot \mathbf{u}) = (km) \cdot \mathbf{u}$, and

M5. for every $\mathbf{u} \in V$, $1 \cdot \mathbf{u} = \mathbf{u}$.

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