UNIVERSITY OF MANITOBA

DATE: February 4, 2010

TERM TEST 1

TITLE PAGE

DEPARTMENT & COURSE NO: MATH 2300

TIME: 75 minutes

EXAMINATION: Linear Algebra II

EXAMINER: Borgersen

NAME: (Print in ink) ______________________________

STUDENT NUMBER: ______________

SIGNATURE: (in ink) ______________________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 75 minute exam. **Please show your work clearly.**

No texts or notes are permitted. No calculators are permitted. Cell phones, electronic translators, and other electronic devices are **not** permitted.

This exam has a title page and 10 pages of questions, including 2 blank pages for rough work and 1 page showing the axioms of a vector space. Please check that you have all the pages. You may remove the blank pages and axiom page if you want, but be careful not to loosen the staple.

The value of each question is indicated beside the statement of the question. The total value of all questions is 70 points.

If you need more scrap paper, use the back of the question pages.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	70	

True or False Questions

1. [20 points] Are the following true or false? (Write "True" or "False" on the line to the right). **These are marked right minus wrong, so if you don't know, don't guess**. Two marks each.

   (a) If S is a subspace of a vector space V, and $\mathbf{0}$ is the zero vector in V, then it must be that $\mathbf{0}$ is also in S.

   (a) ____________________

   (b) There exists a subspace W of $\mathbb{R}^2$ that contains the vectors $(1, 0)$ and $(0, 1)$, and yet $W \neq \mathbb{R}^2$.

   (b) ____________________

   (c) For any matrix A, if the row vectors and the column vectors both form linearly independent sets then A must be square.

   (c) ____________________

   (d) The set V of 2×2 lower triangular matrices is a vector space of dimension 2.

   (d) ____________________

   (e) If two non-zero vectors in $\mathbb{R}^n$ are orthogonal, they must be linearly independent.

   (e) ____________________

   (f) The set $\{1, x^2, 1 + x^3\}$ forms a basis for P_3.

   (f) ____________________

   (g) The set of all vectors in $\mathbb{R}^2$ that are perpendicular to $(1, 2)$ is a vector space using the usual vector addition and scalar multiplication.

   (g) ____________________

   (h) The only subspaces of $\mathbb{R}^3$ are lines and planes that contain the origin.

   (h) ____________________

   (i) The set $V = \{f(x) \in P_3 : f(1) = 1\}$ is a subspace of P_3.

   (i) ____________________

   (j) The set

   $$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 2, \quad a, b, c, d \in \{0, 1\} \right\}$$

   is linearly independent in $M_{2,2}$.

   (j) ____________________

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2. [10 points] Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Prove that every vector $\mathbf{v} \in V$ can be expressed as a linear combination of the elements of S in exactly one way.

3. [10 points] Let V be a vector space, and let W_1 and W_2 be subspaces of V. Let

$$U = \{\mathbf{v} : \mathbf{v} \in W_1 \text{ and } \mathbf{v} \in W_2\}$$

(that is, U is the set of vectors in BOTH W_1 and W_2). Prove that U is a subspace of V as well. Explicitly refer to any and all axioms you use along the way.

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4. Let $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (3, -1, 0) \in \mathbb{R}^3$. Let $\mathbf{w} = (5, 3, 2)$ and $\mathbf{u} = (7, 0, 0)$.

   (a) [4 points] Show that $\mathbf{w} \in \mathrm{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$

   (b) [6 points] Show that $\mathbf{u} \notin \mathrm{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$

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5. Let

$$S = \left\{ \begin{bmatrix} a & b \\ a+b & b \end{bmatrix} : a, b \in \mathbb{R} \right\} \subseteq M_{2,2}.$$

(a) [4 points] Prove that S a subspace of $M_{2,2}$.

(b) [4 points] Find a basis for S, and prove your answer is a basis.

(c) [2 points] What is the dimension of S? $\mathrm{Dim}(S) =$ ____________________

6. [10 points] Show that

$$S = \{x, 2 + x^2, 5 + x\}$$

forms a basis for P_2.

The Axioms of a Vector Space

A set V together with addition "$\oplus$" and scalar multiplication "$\cdot$" is a **Vector Space** if and only if each of the following axioms hold:

A1. for every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \oplus \mathbf{v} \in V$,

A2. for every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$,

A3. for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$,

A4. there exists an element $\mathbf{0} \in V$ such that for every $\mathbf{u} \in V$, $\mathbf{0} \oplus \mathbf{u} = \mathbf{u}$,

A5. for every $\mathbf{u} \in V$, there exists a "$-\mathbf{u}$" $\in V$ such that $\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}$,

M1. for every $\mathbf{u} \in V$ and $k \in \mathbb{R}$, $k \cdot \mathbf{u} \in V$,

M2. for every $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$, $k \cdot (\mathbf{u} \oplus \mathbf{v}) = k \cdot \mathbf{u} \oplus k \cdot \mathbf{v}$,

M3. for every $\mathbf{u} \in V$, and $k, m \in \mathbb{R}$, $(k + m) \cdot \mathbf{u} = k \cdot \mathbf{u} \oplus m \cdot \mathbf{u}$,

M4. for every $\mathbf{u} \in V$, and $k, m \in \mathbb{R}$, $k \cdot (m \cdot \mathbf{u}) = (km) \cdot \mathbf{u}$, and

M5. for every $\mathbf{u} \in V$, $1 \cdot \mathbf{u} = \mathbf{u}$.

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