UNIVERSITY OF MANITOBA

DATE: March 9, 2010

TERM TEST 2 SOLUTIONS

TITLE PAGE

DEPARTMENT & COURSE NO: MATH 2300

TIME: 75 minutes

EXAMINATION: Linear Algebra II

EXAMINER: Borgersen

NAME: (Print in ink) ______________________________

STUDENT NUMBER: ______________

SIGNATURE: (in ink) ______________________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 75 minute exam. **Please show your work clearly.**

No texts or notes are permitted. No calculators are permitted. Cell phones, electronic translators, and other electronic devices are **not** permitted.

This exam has a title page and 11 pages of questions, including 2 blank pages for rough work and 1 page showing the axioms of a vector space. Please check that you have all the pages. You may remove the blank pages and axiom page if you want, but be careful not to loosen the staple.

The value of each question is indicated beside the statement of the question. The total value of all questions is 78 points.

If you need more scrap paper, use the back of the question pages.

Question	Points	Score
1	10	
2	12	
3	10	
4	14	
5	4	
6	6	
7	10	
8	12	
Total:	78	

True or False Questions

1. [10 points] Are the following true or false? (Write "True" or "False" on the line to the right). **These are marked right minus wrong, so if you don't know, don't guess**. One mark each.

   (a) Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T : V \to V$ be an linear operator. Then $T(B) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ must be a basis for V as well.

   (a) False

   (b) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by $T(a, b, c, d) = (a, b, c)$. Then T is an isomorphism.

   (b) False

   (c) Let A be an invertible matrix. Then T_A must be 1-to-1.

   (c) True

   (d) The vector spaces P_4 and $\mathbb{R}^4$ are isomorphic.

   (d) False

   (e) Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects vectors in the xz-plane, and then cuts their lengths in half. Then

   $$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

   (e) True

   (f) Let $T : \mathbb{R}^2 \to \mathbb{R}$ be defined by $T(a, b) = ||(a, b)||$. Then T is an onto linear transformation.

   (f) False

   (g) Let $T : P_2 \to P_1$ be defined as $T(f(x)) = f'(x)$. Then T is a linear transformation.

   (g) True

   (h) Let $T : V \to W$ be a one-to-one linear transformation. Then $T^{-1} : T(V) \to V$ is an isomorphism.

   (h) True

   (i) Let $T : V \to W$ be a one-to-one linear transformation. Then $\text{nullity}(T^{-1}) = 0$.

   (i) True

   (j) Let $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ be a set of vectors in V and let T be a linear transformation $T : V \to W$ be such that $T(S) = \{T(\mathbf{s}_1), \dots, T(\mathbf{s}_n)\}$ is a linearly independent set in W. Then S must have been linearly independent.

   (j) True

Long Answer Questions

2. Let $T : P_2 \to P_1$ be defined by

$$T(a + bx + cx^2) = (a + b)x.$$

(a) [8 points] Show that T is a linear transformation.

Solution: Let $a + bx + cx^2, d + ex + fx^2 \in P_2$, $k \in \mathbb{R}$.

1)

$$\begin{aligned} T((a + bx + cx^2) + (d + ex + fx^2)) &= T((a + d) + (b + e)x + (c + f)x^2) \\ &= ((a + d) + (b + e))x \\ &= (a + d + b + e)x \\ &= (a + b)x + (d + e)x \\ &= T(a + bx + cx^2) + T(d + ex + fx^2). \end{aligned}$$

2)

$$\begin{aligned} T(k(a + bx + cx^2)) &= T(ka + (kb)x + (kc)x^2) \\ &= (ka + kb)x \\ &= k((a + b)x) \\ &= kT(a + bx + cx^2). \end{aligned}$$

Thus by definition, T is a linear transformation.

Reminder: $T : P_2 \to P_1$ is defined by

$$T(a + bx + cx^2) = (a + b)x.$$

(b) [4 points] Find a basis for the range of T.

Solution:

$$\begin{aligned}
T(P_2) &= \{T(\mathbf{v}) : \mathbf{v} \in P_2\} \\
&= \{T(a + bx + cx^2) : a + bx + cx^2 \in P_2\} \\
&= \{T(a + bx + cx^2) : a, b, c \in \mathbb{R}\} \\
&= \{(a + b)x : a, b \in \mathbb{R}\} \\
&= \{ax + bx : a, b \in \mathbb{R}\} \\
&= \text{span}(\{x, x\}) \\
&= \text{span}(\{x\}).
\end{aligned}$$

Thus $\{x\}$ spans the range, and since it is a one-element set, it is certainly linearly independent, and thus forms a basis for the range of T.

---

3. [10 points] Let $T : P_2 \to M_{2,2}$ be defined by

$$T(f(x)) = \begin{bmatrix} f(1) & f(0) \\ f(-1) & -f(0) \end{bmatrix}.$$

Find a set that spans $\ker(T)$, and find $\text{nullity}(T)$.

Solution:

$$\begin{aligned}
\ker(T) &= \left\{ \mathbf{v} \in P_2 : T(\mathbf{v}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ a + bx + cx^2 : T(a + bx + cx^2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ a + bx + cx^2 : \begin{bmatrix} a+b+c & a \\ a-b+c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\
&= \left\{ a + bx + cx^2 : a+b+c = 0,\ a = 0,\ a-b+c = 0,\ -a = 0 \right\} \\
&= \left\{ a + bx + cx^2 : b+c = 0,\ a = 0,\ -b+c = 0 \right\} \\
&= \left\{ bx + cx^2 : c = -b, c = b \right\} \\
&= \left\{ bx + cx^2 : c = b = 0 \right\} \\
&= \{0\}.
\end{aligned}$$

Thus $\{0\}$ spans the kernel, and so the nullity of T is 0.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(1,0) = (1,1) \qquad \text{and} \qquad T(0,1) = (1,-1).$$

(a) [8 points] Find a general formula for $T(x,y)$.

Solution:

$$\begin{aligned} T(x,y) &= T(x(1,0) + y(0,1)) \\ &= T(x(1,0)) + T(y(0,1)) \\ &= xT(1,0) + yT(0,1) \\ &= x(1,1) + y(1,-1) \\ &= (x,x) + (y,-y) \\ &= (x+y, x-y). \end{aligned}$$

(b) [2 points] What is $T(3,5)$?

Solution:

$$T(3,5) = (3+5, 3-5) = (8,-2).$$

(c) [4 points] Find a general formula for $(T \circ T)(x,y)$.

Solution:

$$\begin{aligned} (T \circ T)(x,y) &= T(T(x,y)) \\ &= T(x+y, x-y) \\ &= ((x+y)+(x-y), (x+y)-(x-y)) \\ &= (2x, 2y). \end{aligned}$$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $T(x, y) = (0, 4y + x)$

   (a) [2 points] Is T one-to-one? Justify your answer.

   Solution: No, since $T(0, 0) = T(-4, 1) = (0, 0)$.

   (b) [2 points] Is T onto? Briefly justify your answer.

   Solution: No, since $T(x, y) = (1, 0)$ has no solution (that is, nothing maps onto $(1, 0)$).

6. Let T_A be the linear transformation associated with the matrix

$$A = \begin{bmatrix} 1 & 4 & 5 & 0 & 9 \\ 3 & -2 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 2 & 3 & 5 & 1 & 8 \end{bmatrix}.$$

   (a) [2 points] What is the domain of T_A?

   Solution: Since A is 4×5, $T_A : \mathbb{R}^5 \to \mathbb{R}^4$. So the domain is $\mathbb{R}^5$.

   (b) [2 points] What is the codomain of T_A?

   Solution: As above, the codomain is $\mathbb{R}^4$.

   (c) [2 points] Is T_A one-to-one?

   Solution: Definitely not, since it is mapping a 5 dimensional space into a 4 dimensional space.

General Proofs

7. [10 points] Let $T : V \to W$ be a linear transformation. Prove that the $\ker(T)$ is a subspace of V.

Solution: To show that $\ker(T)$ we must show that it is closed under addition and under scalar multiplication. Let $\mathbf{u}, \mathbf{v} \in \ker(T)$, and let $k \in \mathbb{R}$.

1) We need to check if $\mathbf{u} + \mathbf{v} \in \ker(T)$. To check, take T of it:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W.$$

Thus $\mathbf{u} + \mathbf{v} \in \ker(T)$.

2) We need to check if $k\mathbf{u} \in \ker(T)$. To check, take T of it:

$$T(k\mathbf{u}) = kT(\mathbf{u}) = k\mathbf{0}_W = \mathbf{0}_W.$$

Thus $k\mathbf{u} \in \ker(T)$.

Thus by the subspace theorem, $\ker(T)$ is a subspace of V.

8. [12 points] Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V, and let $T : V \to W$ be a linear transformation. Show that if $T(\mathbf{v}_1) = T(\mathbf{v}_2) = \cdots = T(\mathbf{v}_n) = \mathbf{0}_W$, then T is the zero transformation (that is, for every $\mathbf{v} \in V$, $T(\mathbf{v}) = \mathbf{0}_W$).

Solution: Let $\mathbf{v} \in V$. Then since B is a basis for V, there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n$.

Then,

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n) \\ &= T(c_1\mathbf{v}_1) + \ldots + T(c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + \ldots + c_nT(\mathbf{v}_n) \\ &= c_1\mathbf{0}_W + \ldots + c_n\mathbf{0}_W \\ &= \mathbf{0}_W. \end{aligned}$$

Thus T is the zero transformation.

The Axioms of a Vector Space

A set V together with addition "$\oplus$" and scalar multiplication "$\cdot$" is a **Vector Space** if and only if each of the following axioms hold:

A1. for every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \oplus \mathbf{v} \in V$,

A2. for every $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$,

A3. for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$,

A4. there exists an element $\mathbf{0} \in V$ such that for every $\mathbf{u} \in V$, $\mathbf{0} \oplus \mathbf{u} = \mathbf{u}$,

A5. for every $\mathbf{u} \in V$, there exists a "$-\mathbf{u}$" $\in V$ such that $\mathbf{u} \oplus (-\mathbf{u}) = \mathbf{0}$,

M1. for every $\mathbf{u} \in V$ and $k \in \mathbb{R}$, $k \cdot \mathbf{u} \in V$,

M2. for every $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$, $k \cdot (\mathbf{u} \oplus \mathbf{v}) = k \cdot \mathbf{u} \oplus k \cdot \mathbf{v}$,

M3. for every $\mathbf{u} \in V$, and $k, m \in \mathbb{R}$, $(k + m) \cdot \mathbf{u} = k \cdot \mathbf{u} \oplus m \cdot \mathbf{u}$,

M4. for every $\mathbf{u} \in V$, and $k, m \in \mathbb{R}$, $k \cdot (m \cdot \mathbf{u}) = (km) \cdot \mathbf{u}$, and

M5. for every $\mathbf{u} \in V$, $1 \cdot \mathbf{u} = \mathbf{u}$.

SCRAP PAPER

SCRAP PAPER