UNIVERSITY OF MANITOBA

DATE: May 31, 2010

MIDTERM SOLUTIONS

TITLE PAGE

DEPARTMENT & COURSE NO: MATH 1300

TIME: 1 hour

EXAMINATION: Vector Geometry and Linear Algebra

EXAMINER: Borgersen

NAME: (Print in ink) ________________________________

STUDENT NUMBER: ______________

SIGNATURE: (in ink) ________________________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 1 hour exam. **Please show your work clearly.**

No texts or notes are permitted. No calculators are permitted. Cell phones, electronic translators, and other electronic devices are **not** permitted.

This exam has a title page and 8 pages of questions, including 2 blank pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staples.

The value of each question is indicated beside the statement of the question. The total value of all questions is 60 points.

If you need more scrap paper, use the back of the question pages.

Question	Points	Score
1	8	
2	10	
3	4	
4	4	
5	8	
6	10	
7	6	
8	10	
Total:	60	

Short Answer

1. [8 points] Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}, \; C = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}.$$

In each of the following cases, compute the given expression or briefly explain why the expression cannot be calculated:

(a) AB

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 6 & 8 & 10 \end{bmatrix}.$$

(b) $A + B$

Solution: Can't be done because A and B are different sizes.

(c) $B + 2C^T$

Solution:

$$\begin{aligned} B + 2C^T &= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} + 2 \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}^T \\ &= \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}. \end{aligned}$$

(d) $AB - BA$

Solution: Can't be done because BA can't be done (the number of columns of B does not equal to the number of rows of A).

---

2. Let A be a 3×3 matrix with determinant 5, and let B be a 3×3 matrix with determinant -3. Find the determinant of each of the following (showing all necessary work):

   (a) [3 points] AB^2

   Solution:
   $$\det(AB^2) = \det(ABB) = \det(A)\det(B)\det(B) = (5)(-3)(-3) = 45.$$

   (b) [3 points] $A^{-1}(2B)A^T$

   Solution:
   $$\det(A^{-1}(2B)A^T) = \det(A^{-1})\det(2B)\det(A^T) = \frac{1}{\det(A)} 2^3 \det(B)\det(A) = 8(-3) = -24.$$

   (c) [4 points] $\operatorname{adj}(A)$

   Solution: Since $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$, we have that
   $$\det(\operatorname{adj}(A)) = \det(\det(A)A^{-1}) = (\det(A))^3 \det(A^{-1}) = (\det(A))^3 \frac{1}{\det(A)} = (\det(A))^2 = 5^2 = 25.$$

3. [4 points] Suppose A is a 4×4 invertible matrix.

   (a) What is the reduced row echelon form of A?

   Solution: Since A is invertible, the RREF of A is I_4.

   (b) Find all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

   Solution: Since A^{-1} exists, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

---

4. Consider the system of linear equations

$$\begin{array}{rcrcc} x & + & ay & = & 2-b \\ 4x & + & 2ay & = & 2b \end{array}.$$

This system's augmented matrix is $\left[\begin{array}{cc|c} 1 & a & 2-b \\ 4 & 2a & 2b \end{array}\right]$, which, partially reduced is $\left[\begin{array}{cc|c} 1 & 0 & -2+2b \\ 0 & a & 4-3b \end{array}\right]$.

(a) [2 points] Find all a and b such that the system has no solutions.

Solution: We need the bottom row to be zeros but not in the last column, so we need $a = 0$ and $4 - 3b \neq 0$, that is $a = 0$ and $b \neq \frac{4}{3}$.

(b) [2 points] Find all a and b such that the system has infinitely many solutions.

Solution: We need the bottom row to be all zeros, so we need $a = 0$ and $b = \frac{4}{3}$.

Long Answer

5. [8 points] Let A be some fixed square matrix. Let B and C be two inverses of A. Prove then that $B = C$ (that is, that there is only one inverse of A). Show all necessary steps, and use complete sentences.

Solution: Since B is an inverse of A we know that $AB = BA = I$, and since C is an inverse of A we know that $AC = CA = I$. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

---

6. Let

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ -2 & 2 & 2 \end{bmatrix}.$$

The adjoint of A is shown below partially computed.

(a) [4 points] Enter the two missing numbers in the boxes provided.

$$\mathrm{adj}(A) = \begin{bmatrix} -4 & -4 & 4 \\ -8 & 4 & -4 \\ 4 & -8 & -4 \end{bmatrix}$$

(b) [4 points] Find the determinant of A (using any valid method)

Solution:

$$\begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ -2 & 2 & 2 \end{vmatrix} R_3 \leftarrow R_3 - R_1 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ -4 & 0 & 2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 2 & 2 \\ -4 & 2 \end{vmatrix}$$

$$= (-2)(4+8) = -2(12) = -24.$$

(c) [2 points] Use (a) and (b) above to find A^{-1}.

Solution:

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \frac{1}{-24} \begin{bmatrix} -4 & -4 & 4 \\ -8 & 4 & -4 \\ 4 & -8 & -4 \end{bmatrix}.$$

7. [6 points] Solve the following system using Gauss-Jordan Elimination. No marks will be awarded for any other method.

$$\begin{array}{rcrcrcrcr} x_1 & & & + & 2x_3 & + & x_4 & = & 10 \\ -x_1 & + & x_2 & + & x_3 & - & x_4 & = & -5 \\ x_1 & + & x_2 & + & 5x_3 & + & 2x_4 & = & 21 \\ & & x_2 & + & 3x_3 & & & = & 5 \end{array}$$

Solution:

$$x_1 + 2x_3 + x_4 = 10$$

$$-x_1 + x_2 + x_3 - x_4 = -5$$

$$x_1 + x_2 + 5x_3 + 2x_4 = 21$$

$$x_2 + 3x_3 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ -1 & 1 & 1 & -1 & -5 \\ 1 & 1 & 5 & 2 & 21 \\ 0 & 1 & 3 & 0 & 5 \end{array}\right]$$

$$R_2 \leftarrow R_2 + R_1$$

$$R_3 \leftarrow R_3 - R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 3 & 1 & 11 \\ 0 & 1 & 3 & 0 & 5 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_2$$

$$R_4 \leftarrow R_4 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 10 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_3 = 4$$

$$x_2 + 3x_3 = 5$$

$$x_4 = 6$$

$$0 = 0$$

$$x_1 = 4 - 2r$$

$$x_2 = 5 - 3r$$

$$x_3 = r$$

$$x_4 = 6$$

$$r \in \mathbb{R}$$

---

8. [10 points] Express $A = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$ as a product of elementary matrices. Show all your work.

Solution:

$$\begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$$

$$R_2 \leftarrow \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},$$

and $E_3 E_2 E_1 A = I$.

Since

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ E_3^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix},$$

we have that

$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

SCRAP PAPER

SCRAP PAPER