MATH 1300 Quiz 4 Solutions 10S Term 1

Name: ______________________ Student Number: ______________

Answer all questions and show all your work. (Total Marks: 20).
You have 20 minutes to complete the quiz.

1. Let

$$\mathbf{u} = (2, -1, 0), \quad \mathbf{v} = (-1, 1, 1).$$

[4] (a) Find the area of the parallelogram determined by **u** and **v**.

Solution:

$$\begin{aligned}
\text{Area} &= ||\mathbf{u} \times \mathbf{v}|| \\
&= ||(2, -1, 0) \times (-1, 1, 1)|| \\
&= \left|\left| \det \left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \right) \right|\right| \\
&= \left|\left| \mathbf{i} \det \left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right) - \mathbf{j} \det \left(\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \right) + \mathbf{k} \det \left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right) \right|\right| \\
&= ||\mathbf{i}(-1) - \mathbf{j}(2) + \mathbf{k}(1)|| \\
&= ||(-1, -2, 1)|| \\
&= \sqrt{(-1)^2 + (-2)^2 + (1)^2} = \sqrt{6}.
\end{aligned}$$

[6] (b) All unit vectors parallel to **u**.

Solution: Parallel to **u**:

$$\begin{aligned}
(a, b, c) &= k\mathbf{u} \\
(a, b, c) &= k(2, -1, 0) \\
a &= 2k \\
b &= -k \\
c &= 0.
\end{aligned}$$

Therefore our vector is of the form $(2k, -k, 0)$. Further, because it is to be a unit vector, we know that

$$\begin{aligned} ||(2k,-k,0)|| &= 1 \\ \sqrt{(2k)^2+(-k)^2} &= 1 \\ (2k)^2+(-k)^2 &= 1 \\ 5k^2 &= 1 \\ k^2 &= \frac{1}{5} \\ k &= \pm\frac{1}{\sqrt{5}}. \end{aligned}$$

Therefore the only two unit vectors that are parallel to $\mathbf{u}$ are $\frac{1}{\sqrt{5}}(2,-1,0)$ and $\frac{-1}{\sqrt{5}}(2,-1,0)$.

2. Let ℓ be the line that passes through $P(5,0,3)$ and $Q(6,1,2)$.

[2] (a) Find the parametric form of this line.

Solution:

$$\begin{gathered} \mathbf{x} = \mathbf{p} + t\overrightarrow{PQ} \\ (x,y,z) = (5,0,3) + t((6,1,2)-(5,0,3)) \\ (x,y,z) = (5,0,3) + t((1,1,-1)) \\ x = 5+t, \quad y = 0+t, \quad z = 3-t. \end{gathered}$$

[8] (b) Find the distance between the line ℓ and the point $R(0,1,1)$.

Solution: The line ℓ is given by

$$x = 5+t, \quad y = 0+t, \quad z = 3-t.$$

Pick any point on the line, say, when $t=0$, $P(5,0,3)$. Consider the vector $\overrightarrow{PR} = R - P = (0,1,1) - (5,0,3) = (-5,1,-2)$. Project this vector onto a vector along the line, say $\overrightarrow{PQ} = (1,1,-1)$:

$$\begin{aligned} \text{proj}_{\overrightarrow{PQ}}\overrightarrow{PR} &= \frac{\overrightarrow{PQ} \bullet \overrightarrow{PR}}{||\overrightarrow{PQ}||^2}\overrightarrow{PQ} \\ &= \frac{(1,1,-1)\bullet(-5,1,-2)}{||(1,1,-1)||^2}(1,1,-1) \\ &= \frac{-5+1+2}{1^2+1^2+(-1)^2}(1,1,-1) \\ &= \frac{-2}{3}(1,1,-1). \end{aligned}$$

This vector is along the line, and we know that the vector $\overrightarrow{PR} - \text{proj}_{\overrightarrow{PQ}} \overrightarrow{PR}$ is perpendicular to the line, and has norm equal to the distance:

$$\begin{aligned}
||\overrightarrow{PR} - \text{proj}_{\overrightarrow{PQ}} \overrightarrow{PR}|| &= ||(-5,1,-2) - \frac{-2}{3}(1,1,-1)|| \\
&= ||(-5,1,-2) - (\frac{-2}{3}, \frac{-2}{3}, \frac{2}{3})|| \\
&= ||(-5,1,-2) + (\frac{2}{3}, \frac{2}{3}, \frac{-2}{3})|| \\
&= ||(\frac{-13}{3}, \frac{5}{3}, \frac{-8}{3})|| \\
&= \sqrt{\left(\frac{-13}{3}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{-8}{3}\right)^2} \\
&= \sqrt{\frac{169}{9} + \frac{25}{9} + \frac{64}{9}} \\
&= \sqrt{\frac{258}{9}} \\
&= \frac{\sqrt{258}}{3}.
\end{aligned}$$