Theorem 1. *If a function f is differentiable at $a \in \mathbb{R}$, then f is continuous at a.*

Proof. To show that f is continuous at a, it must be shown that

$$\lim_{x\to a} f(x) = f(a).$$

Since f is differentiable at a,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. So,

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(f(a) + f(x) - f(a)) && \text{(adding and subtracting } f(a)\text{)},\\
&= \lim_{x\to a} f(a) + \lim_{x\to a}(f(x) - f(a)) && \text{(sum law for limits)},\\
&= f(a) + \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) && \text{(multiplying and dividing by } x - a\text{)},\\
&= f(a) + \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x - a) && \left(\text{since } \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \text{ exists}\right),\\
&= f(a) + f'(a) \times 0,\\
&= f(a).
\end{aligned}$$

Therefore f is continuous at a. □

Proof. (ALTERNATIVE PROOF #1)

To show that f is continuous at a, it must be shown that

$$\lim_{x\to a} f(x) = f(a),$$

or equivalently, that

$$\lim_{h\to 0} f(a+h) = f(a).$$

Since f is differentiable at a, $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. So,

$$\begin{aligned}
\lim_{h\to 0} f(a+h) &= \lim_{h\to 0} f(a+h) - f(a) + f(a) && \text{(adding and subtracting } f(a)\text{)},\\
&= \lim_{h\to 0}(f(a+h) - f(a)) + \lim_{h\to 0} f(a) && \text{(sum law for limits)},\\
&= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}(h) + \lim_{h\to 0} f(a) && \text{(multiplying and dividing by } h\text{)},\\
&= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \lim_{h\to 0}(h) + \lim_{h\to 0} f(a) && \left(\text{since } \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}\right),\\
&= f'(a) \times 0 + f(a),\\
&= f(a).
\end{aligned}$$

Therefore f is continuous at a. □

Theorem 2. *For any differentiable function f, and any $c \in \mathbb{R}$,*

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}f(x).$$

Proof. Let $g(x) = cf(x)$. Then,

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(definition of the derivative of } g(x)\text{)},\\
&= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} && \text{(definition of } g(x)\text{)},\\
&= \lim_{h\to 0} c\left(\frac{f(x+h)-f(x)}{h}\right),\\
&= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{(law for limits)},\\
&= cf'(x) && \text{(definition of the derivative of } f(x)\text{)}.
\end{aligned}$$

This proves the theorem. □

Extra Theorem. *For any differentiable function f, and any $c \in \mathbb{R}$,*

$$\frac{d}{dx}\left(\frac{1}{c}f(x)\right) = \frac{1}{c}\left(\frac{d}{dx}f(x)\right).$$

Proof. Let $g(x) = \frac{1}{c}f(x)$. Then,

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(definition of the derivative of } g(x)\text{)},\\
&= \lim_{h\to 0}\frac{\frac{1}{c}f(x+h)-\frac{1}{c}f(x)}{h} && \text{(definition of } g(x)\text{)},\\
&= \lim_{h\to 0}\frac{1}{c}\left(\frac{f(x+h)-f(x)}{h}\right),\\
&= \frac{1}{c}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{(law for limits)},\\
&= \frac{1}{c}f'(x) && \text{(definition of the derivative of } f(x)\text{)}.
\end{aligned}$$

This proves the theorem. □

Theorem 3. *For any differentiable functions f and g,*

$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{d}{dx}f(x)+\frac{d}{dx}g(x).$$

Proof. Let $S(x)=f(x)+g(x)$. Then

$$\begin{aligned}
S'(x) &= \lim_{h\to 0}\frac{S(x+h)-S(x)}{h} && \text{(definition of the derivative of } S(x)\text{)}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} && \text{(definition of } S(x)\text{)}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right) && \text{(reordering and separating the fraction)}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(limit law for sums)}\\
&= f'(x)+g'(x).
\end{aligned}$$

This proves the theorem. □

Extra Theorem. *For any differentiable functions f and g,*

$$\frac{d}{dx}\left(f(x)-g(x)\right)=\frac{d}{dx}f(x)-\frac{d}{dx}g(x).$$

Proof. Let $S(x)=f(x)-g(x)$. Then

$$\begin{aligned}
S'(x) &= \lim_{h\to 0}\frac{S(x+h)-S(x)}{h} && \text{(definition of the derivative of } S(x)\text{)}\\
&= \lim_{h\to 0}\frac{f(x+h)-g(x+h)-f(x)+g(x)}{h} && \text{(definition of } S(x)\text{)}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}-\frac{g(x+h)-g(x)}{h}\right) && \text{(reordering and separating the fraction)}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && \text{(limit law for sums)}\\
&= f'(x)-g'(x).
\end{aligned}$$

This proves the theorem. $\square$

Theorem 4. *For any differentiable functions f and g, $(f(x)g(x))' = f(x)'g(x) + g'(x)f(x)$.*

Proof. Let $S(x) = f(x)g(x)$. Then

$$\begin{aligned}
S'(x) &= \lim_{h\to 0} \frac{S(x+h)-S(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} && \text{(definition of } S(x)\text{)} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \frac{f(x)g(x+h)-f(x)g(x)}{h} \right) && \text{(separating the fraction)} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h)-f(x)g(x)}{h} && \text{(limit law for sums)} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x+h) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} f(x) && \text{(factoring)} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \lim_{h\to 0} g(x+h) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \lim_{h\to 0} f(x) && \text{(limit rule for products)} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} f(x) \\
&= f'(x)g(x) + g'(x)f(x).
\end{aligned}$$

This proves the theorem. □

Extra Theorem. *By the definition of the derivative, show that for any differentiable function f,*

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{f(x)^2}.$$

Proof. Let $S(x) = \frac{1}{f(x)}$. Then

$$\begin{aligned}
S'(x) &= \lim_{h\to 0} \frac{S(x+h) - S(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} && \text{(definition of } S(x)\text{)} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{f(x+h)}\left(\frac{f(x)}{f(x)}\right) - \frac{1}{f(x)}\left(\frac{f(x+h)}{f(x+h)}\right)\right) && \text{(getting a common denominator)} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x) - f(x+h)}{f(x)f(x+h)}\right) \\
&= \frac{1}{f(x)} \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x) - f(x+h)}{f(x+h)}\right) && \text{(}f(x)\text{ doesn't depend on }h\text{)} \\
&= \frac{1}{f(x)} \lim_{h\to 0}\left(\frac{1}{f(x+h)}\right)\left(\frac{f(x) - f(x+h)}{h}\right) && \text{(reordering and grouping)} \\
&= \frac{-1}{f(x)} \lim_{h\to 0}\left(\frac{1}{f(x+h)}\right)\left(\frac{f(x+h) - f(x)}{h}\right) && \text{(factor out } -1\text{)} \\
&= \frac{-1}{f(x)} \lim_{h\to 0}\left(\frac{1}{f(x+h)}\right) \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) && \text{(limit rule)} \\
&= \frac{-1}{f(x)}\left(\frac{1}{f(x)}\right)(f'(x)) && \text{(evaluating the limits)} \\
&= \frac{-f'(x)}{f(x)^2}.
\end{aligned}$$

This proves the theorem. □

Theorem 5. $(\sin(x))' = \cos(x)$.

Proof. For the proof, two claims are needed:

Claim 1. $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$.

Claim 2. $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$.

Once these claims are established, the derivative of $\sin(x)$ is found as follows:

$$\begin{aligned}
(\sin(x))' &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h)-\sin(x)+\cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \left(\sin(x)\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h} \right) \\
&= \lim_{h\to 0} \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \lim_{h\to 0} \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x) \times 0 + \cos(x) \times 1 && \text{(by claims 1 and 2)} \\
&= \cos(x).
\end{aligned}$$

This proves the theorem. $\square$

Theorem 6. *If for a differentiable function f on some interval (a, b), $f'(x) = 0$, then f is constant on (a, b).*

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But we assumed that $f'(x) = 0$ for all x in (a, b). Therefore, $f(x_2) - f(x_1) = 0$, and so $f(x_2) = f(x_1)$. Therefore, for *any* two numbers x_1, x_2 in (a, b), $f(x_1) = f(x_2)$, and so the function is constant on (a, b). This proves the theorem. $\square$

Theorem 7. *If for a differentiable function f on some interval I, $f'(x) > 0$, then f is (strictly) increasing on I.*

Proof. Let x_1 and x_2 be any two numbers in the interval, $x_1 < x_2$. According to the definition of an increasing function, we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on the interval (specifically on $[x_1, x_2]$, and so by the Mean Value Theorem, there is a number c, $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) > 0$, the fraction on the right is positive. Since $x_2 > x_1$, $x_2 - x_1 \geq 0$, and so $f(x_2) - f(x_1) > 0$ too. Thus $f(x_2) > f(x_1)$, and so the function is increasing. This proves the theorem. $\square$

Theorem 8. *If for a differentiable function f on some interval I, $f'(x) < 0$, then f is (strictly) decreasing on I.*

Proof. Let x_1 and x_2 be any two numbers in the interval, $x_1 < x_2$. According to the definition of a decreasing function, we have to show that $f(x_1) > f(x_2)$.

Because we are given that $f'(x) < 0$, we know that f is differentiable on the interval (and specifically on $[x_1, x_2]$, and so by the Mean Value Theorem, there is a number c, $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$, the fraction on the right is negative. Since $x_2 > x_1$, $x_2 - x_1 \geq 0$, and so $f(x_2) - f(x_1) < 0$. Thus $f(x_1) > f(x_2)$, and so the function is decreasing. This proves the theorem. ☐

Proof. (ALTERNATE PROOF #1)

Let $x_1 < x_2$ be two values in I. Since f is differentiable on I, f is continuous on $[x_1, x_2]$, and is differentiable on (x_1, x_2). Therefore, by the Mean Value Theorem, there exists c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Then,

$$\begin{aligned} f(x_2) &= f(x_2) - f(x_1) + f(x_1) \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_2 - x_1) + f(x_1) \\ &= f'(c)(x_2 - x_1) + f(x_1) \\ &< f(x_1). && \text{(since } x_2 - x_1 > 0 \text{ and } f'(c) < 0\text{)} \end{aligned}$$

Therefore f is decreasing. This proves the theorem. ☐

Proof. (ALTERNATE PROOF #2)

Let $x_1 < x_2$ be two values in I. Since f is differentiable on I, f is continuous on $[x_1, x_2]$, and is differentiable on (x_1, x_2). Therefore, by the Mean Value Theorem, there exists c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Assume that $f(x_1) < f(x_2)$. Then, since $x_1 < x_2$, it follows that $f'(c) > 0$, a contradiction. Therefore, it must be that $f(x_1) > f(x_2)$, proving f is decreasing. This proves the theorem. ☐