

Theorem 1. *If a function f is differentiable at $a \in \mathbb{R}$, then f is continuous at a .*

Proof. To show that f is continuous at a , it must be shown that

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Since f is differentiable at a ,

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. So,

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (f(a) + f(x) - f(a)) && \text{(adding and subtracting } f(a)\text{),} \\ &= \lim_{x \rightarrow a} f(a) + \lim_{x \rightarrow a} (f(x) - f(a)) && \text{(sum law for limits),} \\ &= f(a) + \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} (x - a) && \text{(multiplying and dividing by } x - a\text{),} \\ &= f(a) + \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \lim_{x \rightarrow a} (x - a) && \text{(since } \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \text{ exists),} \\ &= f(a) + f'(a) \times 0, \\ &= f(a). \end{aligned}$$

Therefore f is continuous at a . □

Theorem 2. For any differentiable function f , and any $c \in \mathbb{R}$,

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x).$$

Proof. Let $g(x) = cf(x)$. Then,

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} && \text{(definition of the derivative of } g(x)\text{),} \\ &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} && \text{(definition of } g(x)\text{),} \\ &= \lim_{h \rightarrow 0} c \left(\frac{f(x+h) - f(x)}{h} \right), \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{(law for limits),} \\ &= cf'(x) && \text{(definition of the derivative of } f(x)\text{).} \end{aligned}$$

This proves the theorem. □

Theorem 3. For any differentiable functions f and g ,

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Proof. Let $S(x) = f(x) + g(x)$. Then

$$\begin{aligned} S'(x) &= \lim_{h \rightarrow 0} \frac{S(x+h) - S(x)}{h} && \text{(definition of the derivative of } S(x)) \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} && \text{(definition of } S(x)) \\ &= \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) && \text{(reordering and separating the fraction)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} && \text{(limit law for sums)} \\ &= f'(x) + g'(x). \end{aligned}$$

This proves the theorem. □

Theorem 4. For any differentiable functions f and g , $(f(x)g(x))' = f(x)'g(x) + g'(x)f(x)$.

Proof. Let $S(x) = f(x)g(x)$. Then

$$\begin{aligned}
 S'(x) &= \lim_{h \rightarrow 0} \frac{S(x+h) - S(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} && \text{(definition of } S(x)) \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h} \right) && \text{(separating the fraction)} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \rightarrow 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} && \text{(limit law for sums)} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} f(x) && \text{(factoring)} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \lim_{h \rightarrow 0} g(x+h) + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \lim_{h \rightarrow 0} f(x) && \text{(limit rule for products)} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} g(x) + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} f(x) \\
 &= f'(x)g(x) + g'(x)f(x).
 \end{aligned}$$

This proves the theorem. □

Theorem 5. $(\sin(x))' = \cos(x)$.

Proof. For the proof, two claims are needed:

Claim 1. $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$.

Claim 2. $\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$.

Once these claims are established, the derivative of $\sin(x)$ is found as follows:

$$\begin{aligned}
 (\sin(x))' &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) - \sin(x) + \cos(x)\sin(h)}{h} \\
 &= \lim_{h \rightarrow 0} \left(\sin(x) \frac{\cos(h) - 1}{h} + \cos(x) \frac{\sin(h)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \lim_{h \rightarrow 0} \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
 &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
 &= \sin(x) \times 0 + \cos(x) \times 1 && \text{(by claims 1 and 2)} \\
 &= \cos(x).
 \end{aligned}$$

This proves the theorem. □

Theorem 6. *If for a differentiable function f on some interval (a, b) , $f'(x) = 0$, then f is constant on (a, b) .*

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b) , it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But we assumed that $f'(x) = 0$ for all x in (a, b) . Therefore, $f(x_2) - f(x_1) = 0$, and so $f(x_2) = f(x_1)$. Therefore, for *any* two numbers x_1, x_2 in (a, b) , $f(x_1) = f(x_2)$, and so the function is constant on (a, b) . This proves the theorem. \square

Theorem 7. *If for a differentiable function f on some interval I , $f'(x) > 0$, then f is (strictly) increasing on I .*

Proof. Let x_1 and x_2 be any two numbers in the interval, $x_1 < x_2$. According to the definition of an increasing function, we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on the interval (specifically on $[x_1, x_2]$), and so by the Mean Value Theorem, there is a number c , $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) > 0$, the fraction on the right is positive. Since $x_2 > x_1$, $x_2 - x_1 \geq 0$, and so $f(x_2) - f(x_1) > 0$ too. Thus $f(x_2) > f(x_1)$, and so the function is increasing. This proves the theorem. \square

Theorem 8. *If for a differentiable function f on some interval I , $f'(x) < 0$, then f is (strictly) decreasing on I .*

Proof. Let x_1 and x_2 be any two numbers in the interval, $x_1 < x_2$. According to the definition of a decreasing function, we have to show that $f(x_1) > f(x_2)$.

Because we are given that $f'(x) < 0$, we know that f is differentiable on the interval (and specifically on $[x_1, x_2]$), and so by the Mean Value Theorem, there is a number c , $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$, the fraction on the right is negative. Since $x_2 > x_1$, $x_2 - x_1 \geq 0$, and so $f(x_2) - f(x_1) < 0$. Thus $f(x_1) > f(x_2)$, and so the function is decreasing. This proves the theorem. \square
