Question. *Find*

$$\lim_{t\to 0} \frac{1}{\sqrt{1+t}} - \frac{1}{t}.$$

Solution.

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{\sqrt{1+t}} - \frac{1}{t} &= \lim_{t\to 0} \frac{t}{t\sqrt{1+t}} - \frac{\sqrt{1+t}}{t\sqrt{1+t}} \\
&= \lim_{t\to 0} \frac{t-\sqrt{1+t}}{t\sqrt{1+t}} \\
&= \left[\frac{-1}{0}\right].
\end{aligned}$$

Therefore we check left and right limits to determine if this is equal to positive or negative infinity:

$$\lim_{t\to 0^-} \frac{t-\sqrt{1+t}}{t\sqrt{1+t}} = \frac{-1}{(\text{small -ve})(1)} = \infty$$

and

$$\lim_{t\to 0^+} \frac{t-\sqrt{1+t}}{t\sqrt{1+t}} = \frac{-1}{(\text{small +ve})(1)} = -\infty.$$

Therefore the general limit does not exist.

Question. *Differentiate y, but do not simplify:*

$$y = \sqrt{3x} + x^4 + \frac{1}{3\sqrt{x}} - 5 + \frac{1}{\pi}.$$

Solution. Find all derivatives separately:

$$\frac{d}{dx}\left(\sqrt{3x}\right) = \sqrt{3}\frac{d}{dx}x^{1/2} = \sqrt{3}\left(\frac{1}{2}\right)x^{-1/2} = \frac{\sqrt{3}}{2\sqrt{x}}.$$

$$\frac{d}{dx}x^4 = 4x^3$$

$$\frac{d}{dx}\frac{1}{3\sqrt{x}} = \left(\frac{1}{3}\right)\frac{d}{dx}x^{-1/2} = \frac{1}{3}\left(\frac{-1}{2}\right)x^{-3/2} = \frac{-1}{6\sqrt{x^3}}.$$

$$\frac{d}{dx}(-5) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\pi}\right) = 0$$

Therefore,

$$y' = \frac{\sqrt{3}}{2\sqrt{x}} + 4x^3 + \frac{-1}{6\sqrt{x^3}}.$$

Question. *Differentiate y, but do not simplify:*

$$y = \sqrt{3x} + x^4 + \frac{3}{\sqrt{3x}} - 5 + \frac{1}{\pi}.$$

Solution. Find all derivatives separately:

$$\frac{d}{dx}\left(\sqrt{3x}\right) = \sqrt{3}\frac{d}{dx}x^{1/2} = \sqrt{3}\left(\frac{1}{2}\right)x^{-1/2} = \frac{\sqrt{3}}{2\sqrt{x}}.$$

$$\frac{d}{dx}x^4 = 4x^3$$

$$\frac{d}{dx}\frac{3}{\sqrt{3x}} = 3\frac{d}{dx}(3x)^{-1/2} = 3\left(\frac{-1}{2}\right)(3x)^{-3/2}(3) = \frac{-9}{2\sqrt{(3x)^3}}.$$

$$\frac{d}{dx}(-5) = 0$$

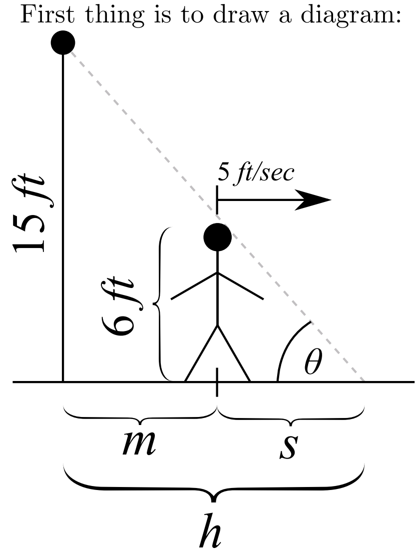
$$\frac{d}{dx}\left(\frac{1}{\pi}\right) = 0$$

Therefore,

$$y' = \frac{\sqrt{3}}{2\sqrt{x}} + 4x^3 + \frac{-9}{2\sqrt{(3x)^3}}.$$

Question. *A street light is mounted at the top of a 15 ft. tall pole. A man 6 ft. tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft. from the pole?*

Solution. First thing is to draw a diagram:



I have chosen some variables in my diagram, m for the distance from the pole to the man, s for the length of the man's shadow, h for the distance from the pole to the head of the shadow, and θ for the angle the light makes with the ground.

Each of the variables m, s, h, and θ are changing over time. So what do we know?

- $\frac{dm}{dt} = 5 ft/sec$
- $m + s = h$
- We want to find $\frac{dh}{dt}$.
- Using similar triangles, I know that $\tan\theta = \frac{15}{h} = \frac{6}{s}$.
- Pythagoras does give us a relationship, but it won't be needed at all, since we are never interested in the distance from the light source to the tip of the shadow's head (which would be the hypotenuse).

So the only real formula I have is

$$m + s = h.$$

Using the similar triangles fact above, I know that

$$s = \frac{6h}{15} = \frac{2h}{5}.$$

Therefore, the equation above becomes

$$m + s = h$$

$$m + (\frac{2}{5}h) = h$$
$$m = h - \frac{2}{5}h$$
$$m = \frac{3}{5}h.$$

Now, taking derivatives of both sides with respect to time, I get:

$$\frac{d}{dt}(m) = \frac{d}{dt}\left(\frac{3}{5}h\right)$$
$$\frac{dm}{dt} = \frac{3}{5}\frac{dh}{dt}.$$

Therefore,

$$\frac{dh}{dt} = \frac{5}{3}\frac{dm}{dt}.$$

Since $\frac{dm}{dt} = 5$, we have that

$$\frac{dh}{dt} = \frac{5}{3}(5) = \frac{25}{3} ft/sec.$$

Question. *Find the critical numbers of*

$$f(x) = \frac{x}{x^2+1} \qquad 0 \le x \le 2.$$

Solution. The critical numbers of a function are the x values at which the derivative either does not exist, or does exist and has value zero. First, find the derivative:

$$\begin{aligned}
f(x) &= x(x^2+1)^{-1} \\
f'(x) &= x(-1)(x^2+1)^{-2}(2x) + 1(x^2+1)^{-1} \\
&= \frac{-2x^2}{(x^2+1)^2} + \frac{1}{x^2+1} \\
&= \frac{-2x^2}{(x^2+1)^2} + \frac{x^2+1}{(x^2+1)^2} \\
&= \frac{-2x^2+x^2+1}{(x^2+1)^2} \\
&= \frac{-x^2+1}{(x^2+1)^2}.
\end{aligned}$$

This derivative is undefined if and only if $(x^2+1)^2 = 0$, which never happens. The derivative is equal to zero if and only if

$$\begin{aligned}
-x^2+1 &= 0 \\
-x^2 &= -1 \\
x^2 &= 1 \\
x &= \pm 1.
\end{aligned}$$

Therefore 1 and -1 are the only critical numbers. Therefore, the critical points are:

$$\left(1, \frac{1}{2}\right) \qquad \left(-1, \frac{-1}{2}\right).$$

Question. *Find the critical numbers of*

$$f(t) = t\sqrt{4-t^2} \qquad -1 \le t \le 2.$$

Solution. The critical numbers of f are the t values at which the derivative either does not exist, or does exist and has value zero. First, find the derivative:

$$\begin{aligned} f(x) &= t(4-t^2)^{\frac{1}{2}} \\ f'(x) &= t(\frac{1}{2})(4-t^2)^{-\frac{1}{2}}(-2t) + (4-t^2)^{\frac{1}{2}} \\ &= \frac{-t^2}{\sqrt{4-t^2}} + \sqrt{4-t^2} \\ &= \frac{-t^2}{\sqrt{4-t^2}} + \frac{4-t^2}{\sqrt{4-t^2}} \\ &= \frac{-2t^2+4}{\sqrt{4-t^2}} \end{aligned}$$

This derivative is undefined if and only if $\sqrt{4-t^2} = 0$, which happens exactly when $t = \pm 2$. Therefore 2 and -2 are critical numbers. The derivative is equal to zero if and only if

$$\begin{aligned} -2t^2 + 4 &= 0 \\ -2t^2 &= -4 \\ t^2 &= 2 \\ t &= \pm\sqrt{2}. \end{aligned}$$

Therefore $\sqrt{2}$ and $-\sqrt{2}$ are also critical numbers. Therefore the critical points are:

$$(2,0) \qquad (-2,0)$$

$$(\sqrt{2},2) \qquad (-\sqrt{2},-2).$$

Question. *A rectangular storage container with an open top is to have a volume of 10* m^3*. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.*

Solution. Let b denote the length of the base, h the length of the height, w the length of the width, V the volume, S the surface area of the box, and C the cost of the box. The volume of a rectangular box is $V = b \times w \times h$, and we want $V = 10$. Therefore, we can solve for one variable in terms of the others:

$$h = \frac{10}{b \times w}.$$

We know that $b = 2 \times w$. Therefore

$$h = \frac{10}{2w^2}.$$

The surface area of the bottom is $b \times w$, the surface area of the long sides are $b \times h$, and the area of the short sides are $w \times h$. The total surface area is

$$S = b \times w + 2b \times h + 2w \times h.$$

The cost function is \$6 per square meter. All the lengths are in meters, and so

$$C = 6 \times S = 6b \times w + 12b \times h + 12w \times h.$$

This is the function to minimize.

$$\begin{aligned} C &= 6 \times S \\ &= 6(b \times w + 2b \times h + 2w \times h) \\ &= 6(2 \times w \times w + 4 \times w \times h + 2w \times h) \\ &= 6w(2w + 4h + 2h) \\ &= 6w(2w + 6h) \\ &= 12w(w + 3h) \\ &= 12w\left(w + 3\frac{10}{2w^2}\right) \\ &= 12\left(w^2 + \frac{15}{w}\right) \\ &= 12w^2 + \frac{180}{w} \\ &= 12w^2 + \frac{180}{w} \end{aligned}$$

Question. *A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?*

Solution. Let x be the length of the side that is bend into a square. Then the area of the square is $(x/4)^2$. In the triangle, each side has length $\ell = (10-x)/3$, and each angle is $\pi/3$. Doing some basic trig then, the area of the triangle is $\frac{\sqrt{3}}{4}\ell^2$. So the total area enclosed is

$$A(x) = (x/4)^2 + \frac{\sqrt{3}}{4}\frac{(10-x)^2}{9}.$$

Simplifying,

$$\begin{aligned}
A(x) &= \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\frac{(10-x)^2}{9} \\
&= \frac{x^2}{16} + \frac{\sqrt{3}}{36}(100-20x+x^2) \\
&= \frac{1}{4}\left(\frac{x^2}{4} + \frac{\sqrt{3}}{9}(100-20x+x^2)\right) \\
&= \frac{1}{4}\left(\frac{9x^2}{36} + \frac{4\sqrt{3}}{36}(100-20x+x^2)\right) \\
&= \frac{1}{4\times 36}\left(9x^2 + 4\sqrt{3}(100-20x+x^2)\right) \\
&= \frac{1}{4\times 36}\left(9x^2 + 400\sqrt{3} - 80x\sqrt{3} + 4x^2\sqrt{3}\right) \\
&= \frac{1}{4\times 36}\left((9+4\sqrt{3})x^2 - 80x\sqrt{3} + 400\sqrt{3}\right).
\end{aligned}$$

The domain of x is $[0, 10]$, covering the two extremes: either bending the entire wire into a square (that is, $x = 10$), or the entire wire into an equilateral triangle (that is, $x = 0$). Now, taking the derivative,

$$\begin{aligned}
A'(x) &= \frac{1}{4\times 36}\left((18+8\sqrt{3})x - 80\sqrt{3}\right) \\
&= \frac{18+8\sqrt{3}}{4\times 36}x - \frac{80\sqrt{3}}{4\times 36} \\
&= \frac{9+4\sqrt{3}}{2\times 36}x - \frac{5\sqrt{3}}{9}.
\end{aligned}$$

The critical numbers of $A(x)$ are when $A'(x) = 0$, that is,

$$\begin{aligned}
& A'(x) = 0 \\
\iff & \frac{9+4\sqrt{3}}{2\times 36}x - \frac{5\sqrt{3}}{9} = 0 \\
\iff & \frac{9+4\sqrt{3}}{2\times 36}x = \frac{5\sqrt{3}}{9} \\
\iff & x = \frac{5\sqrt{3}}{9}\left(\frac{2\times 36}{9+4\sqrt{3}}\right).
\end{aligned}$$

Simplifying,

$$\begin{aligned}
x &= \frac{5\sqrt{3}}{1}\left(\frac{8}{9+4\sqrt{3}}\right) \\
&= \frac{40\sqrt{3}}{9+4\sqrt{3}} \\
&= \frac{40\sqrt{3}}{9+4\sqrt{3}}\left(\frac{9-4\sqrt{3}}{9-4\sqrt{3}}\right) \\
&= \frac{40\sqrt{3}}{9+4\sqrt{3}}\left(\frac{9-4\sqrt{3}}{9-4\sqrt{3}}\right) \\
&= \frac{360\sqrt{3}-160(3)}{81-16(3)} \\
&= \frac{360\sqrt{3}-160(3)}{33} \\
&= \frac{120\sqrt{3}-160}{11} \\
&\approx 4.349645173.
\end{aligned}$$

Therefore, the only critical point is approximately 4.349645173. The max and mins of $A(x)$ can only occur at this point, or at the end points of the domain.

x	$A(x)$
0	$\frac{1}{4\times 36}(400\sqrt{3}) = \frac{25\sqrt{3}}{9} \approx 4.811252243.$
4.349645173	$\approx 2.718528233.$
10	6.25.

(a) The maximum area is found at $x = 10$, that is, don't cut anything and just turn the entire cord into a square. At this point, at area of $6.25m^2$.

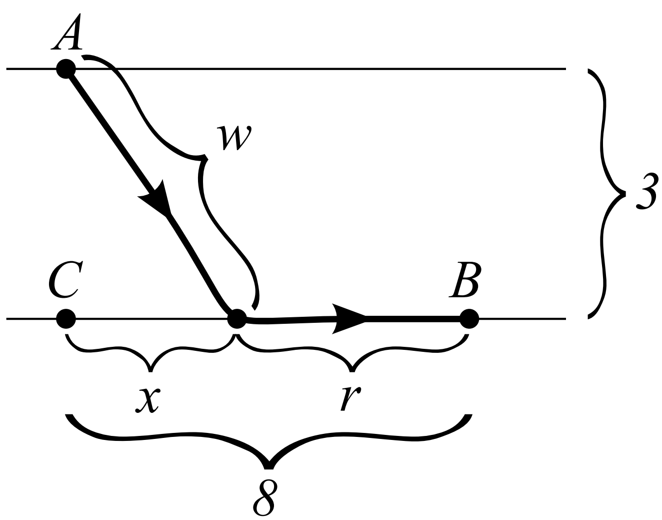
(b) The minimum area is found at $x \approx 4.349645173$, where the total area is $\approx 2.718528233m^2$.

Note: I am a little worried about this solution since the numbers are so ugly, and so I may have made a mistake right near the beginning messing everything up. Let me know if you see any mistakes.

Question. *A man launches his boat from point A on a bank of a straight river, 3km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point C, and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row at 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows).*

Solution.

Here is the diagram of what is going on:



We are interested in minimizing the total amount of time. To travel across the water, the man will go 6 km/hr, for a total of w kilometers. If this will take t_1 hours, then $6t_1 = w$. Therefore, $t_1 = \frac{w}{6}$. To travel across the land, the man will go 8 km/hr, for a total of r kilometers. If this will take t_2 hours, then $8t_2 = r$, or in other words, $t_2 = \frac{r}{8}$. Therefore, the total amount of time this will take is

$$T = t_1 + t_2 = \frac{w}{6} + \frac{r}{8}.$$

This is the function we want to minimize.

Our goal now is to turn this into a function of one variable. We know two relationships between the variables involved:

$$x + r = 8 \qquad x^2 + 3^2 = w^2$$

Solution 1. I could replace all the w's with some function of r, given by the second relationship above: $x^2+3^2 = w^2 \Rightarrow w = \sqrt{x^2+9}$. Further, $x = 8 - r$. So, I get:

$$w = \sqrt{x^2+9} = \sqrt{(8-r)^2+9}$$
$$= \sqrt{r^2-16r+64+9} = \sqrt{r^2-16r+73}.$$

Then, the time function becomes

$$T = \frac{w}{6} + \frac{r}{8} = \frac{\sqrt{r^2-16r+73}}{6} + \frac{r}{8}.$$

Now T is completely a function of r.

Simplifying T, we get:

$$\begin{aligned} T &= \frac{\sqrt{r^2-16r+73}}{6} + \frac{r}{8} \\ &= \frac{\sqrt{r^2-16r+73}}{6}\left(\frac{4}{4}\right) + \frac{r}{8}\left(\frac{3}{3}\right) \\ &= \frac{4\sqrt{r^2-16r+73}}{24} + \frac{3r}{24} \\ &= \frac{4\sqrt{r^2-16r+73}+3r}{24} \\ &= \frac{1}{24}(4\sqrt{r^2-16r+73}+3r) \\ &= \frac{1}{24}(4(r^2-16r+73)^{1/2}+3r). \end{aligned}$$

Note that this function is only valid on a certain domain. Certainly, r, the distance the man runs, can be at the very least 0 if he boats directly to B, while the maximum it can be would be 8. Therefore, the domain of T is $[0, 8]$.

Finding the critical numbers of T will allow us to find the position where the absolute minimum is achieved. To do so, we find the derivative as follows:

$$\begin{aligned} \frac{dT}{dr} &= \frac{1}{24}(4(\frac{1}{2})(r^2-16r+73)^{-1/2}(2r-16)+3) \\ &= \frac{1}{24}(2(r^2-16r+73)^{-1/2}(2r-16)+3) \\ &= \frac{1}{24}(\frac{2(2r-16)}{(r^2-16r+73)^{1/2}}+3) \\ &= \frac{1}{24}(\frac{4r-32}{\sqrt{r^2-16r+73}}+3). \end{aligned}$$

The critical numbers of T are the zeros of the derivative:

$$\frac{dT}{dr} = 0 \qquad \Leftarrow$$

$$\begin{aligned} &\iff \frac{4r-32}{\sqrt{r^2-16r+73}} + 3 = 0 \\ &\iff \frac{4r-32}{\sqrt{r^2-16r+73}} = -3 \\ &\iff 4r-32 = -3\sqrt{r^2-16r+73} \\ &\iff \frac{4r-32}{-3} = \sqrt{r^2-16r+73} \\ &\iff \left(\frac{4r-32}{-3}\right)^2 = r^2-16r+73 \\ &\iff \frac{16r^2-256r+1024}{9} = r^2-16r+73 \end{aligned}$$

$$\begin{aligned}
&\iff \frac{16}{9}r^2 - \frac{256}{9}r + \frac{1024}{9} = r^2 - 16r + 73 \\
&\iff \frac{16}{9}r^2 - r^2 - \frac{256}{9}r + 16r + \frac{1024}{9} - 73 = 0 \\
&\iff \frac{16}{9}r^2 - \frac{9}{9}r^2 - \frac{256}{9}r + 16\left(\frac{9}{9}\right)r + \frac{1024}{9} - 73\left(\frac{9}{9}\right) = 0 \\
&\iff \frac{7}{9}r^2 - \frac{256}{9}r + \frac{144}{9}r + \frac{1024}{9} - \frac{657}{9} = 0 \\
&\iff \frac{7}{9}r^2 - \frac{112}{9}r + \frac{367}{9} = 0 \\
&\iff \frac{1}{9}(7r^2 - 112r + 367) = 0 \\
&\iff 7r^2 - 112r + 367 = 0 \\
&\iff r = \frac{112 \pm \sqrt{(-112)^2 - 4(7)(367)}}{2(7)} \\
&\iff r = \frac{112 \pm \sqrt{12544 - 10276}}{14} \\
&\iff r = \frac{112 \pm \sqrt{2268}}{14} \\
&\iff r = \frac{112 \pm \sqrt{4 \times 567}}{14} \\
&\iff r = \frac{112 \pm 2\sqrt{567}}{14} \\
&\iff r = \frac{56 \pm \sqrt{81 \times 7}}{7} \\
&\iff r = \frac{56 \pm 9\sqrt{7}}{7}.
\end{aligned}$$

Approximately, these solutions are

$$r = \frac{56 + 9\sqrt{7}}{7} \approx \frac{79.811}{7} \approx 11.401$$

and

$$r = \frac{56 - 9\sqrt{7}}{7} \approx \frac{32.188}{7} \approx 4.598.$$

Note that the first critical number is outside of our domain, and so it does not need to be considered.

We now need to find the sign of T' to the left and to the right of the critical number:

$$T' = \frac{1}{24}\left(\frac{4r - 32}{\sqrt{r^2 - 16r + 73}} + 3\right)$$

$$\begin{aligned}
T'(0) &= \frac{1}{24}\left(\frac{4(0) - 32}{\sqrt{(0)^2 - 16(0) + 73}} + 3\right) \\
&= \frac{1}{24}\left(\frac{-32}{\sqrt{73}} + 3\right) \\
&\approx \frac{1}{24}(-3.745 + 3) < 0. \\
T'(8) &= \frac{1}{24}\left(\frac{4(8) - 32}{\sqrt{(8)^2 - 16(8) + 73}} + 3\right) \\
&= \frac{1}{24}\left(\frac{0}{\sqrt{9}} + 3\right) \\
&= \frac{1}{24}(0 + 3) > 0.
\end{aligned}$$

		$\frac{56-9\sqrt{7}}{7}$	
T'	$\ominus$	0	$\oplus$

Therefore, at $r = \frac{56-9\sqrt{7}}{7}$, T achieves it's absolute minimum. To answer the question then, we need to say where the man should land. This is best described using x.

$$\begin{aligned}
x &= 8 - r \\
&= 8 - \frac{56 - 9\sqrt{7}}{7} \\
&= \frac{56}{7} - \frac{56 - 9\sqrt{7}}{7} \\
&= \frac{56 - 56 + 9\sqrt{7}}{7} \\
&= \frac{9\sqrt{7}}{7} \\
&= \frac{9}{\sqrt{7}}.
\end{aligned}$$

Therefore the man should land his boat $\frac{9}{\sqrt{7}}$ kilometers away from point C.