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# Graph Reconstruction

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# Abstract

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The concept of a graph is one of the most basic and readily understood mathematical concepts, and the **Reconstruction Conjecture** is one of the most engaging problems under the domain of Graph Theory. The conjecture proposes that every graph with at least three vertices can be uniquely reconstructed given the multiset of subgraphs produced by deleting each vertex of the original graph one by one. This conjecture has been proven true for several infinite classes of graphs, but the general case remains unsolved. In this talk, I will outline the problem, and introduce some of the most well known results.

# Graph Theory

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The study of Graphs ...

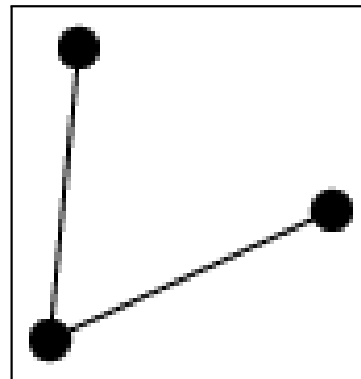
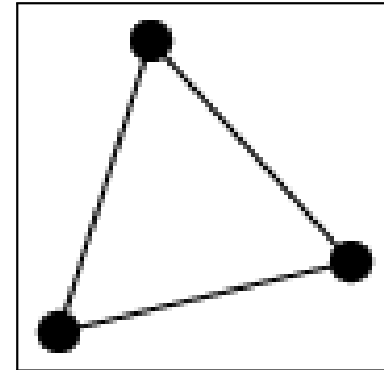
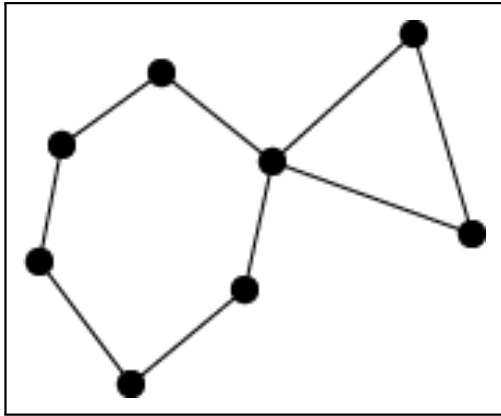
# Graph Theory

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The study of Graphs ... so what's a graph?

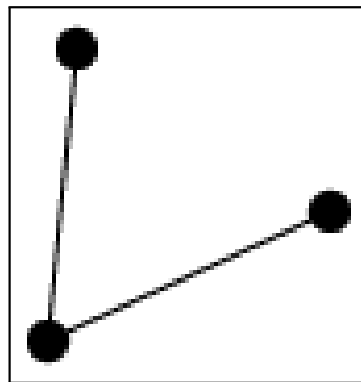
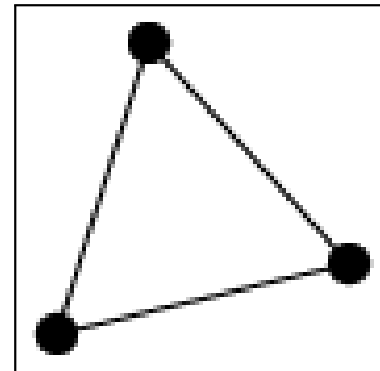
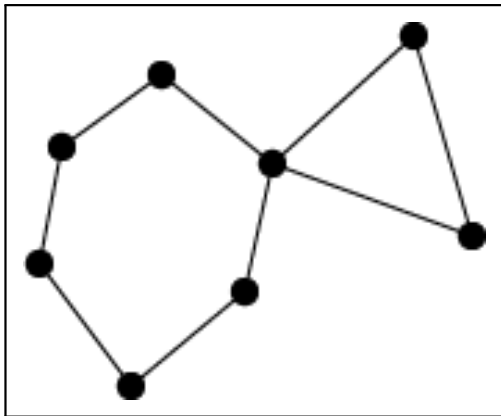
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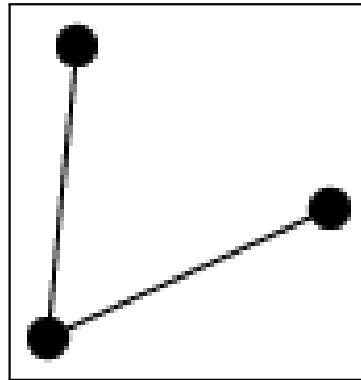
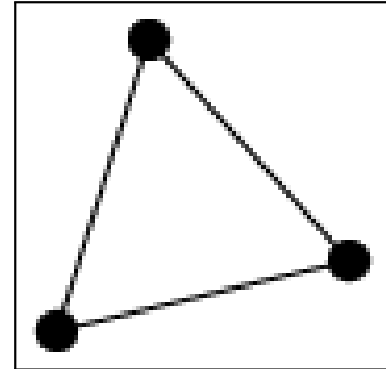
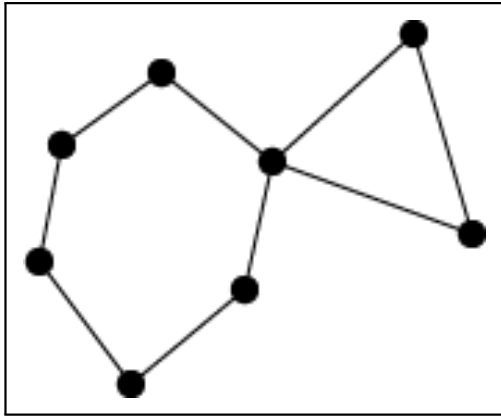
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- $V(G)$  = the set of vertices of  $G$ .
- $E(G)$  = the set of edges of  $G$ .

# Definitions

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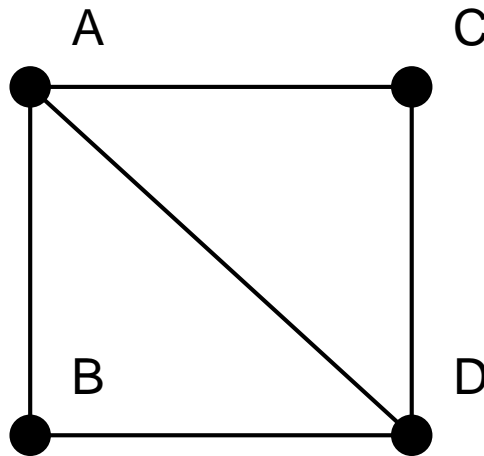
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- $K_n$  denotes the **complete** graph on  $n$  vertices (every possible edge).
- A **card** of  $G$  is an unlabelled graph formed by deleting 1 vertex and all edges attached to it.
- The **deck** of  $G$ ,  $\mathcal{D}(G)$ , is the collection of all  $G$ 's cards. Note this is in general a multiset.

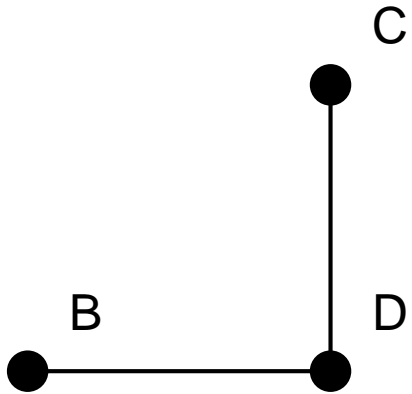
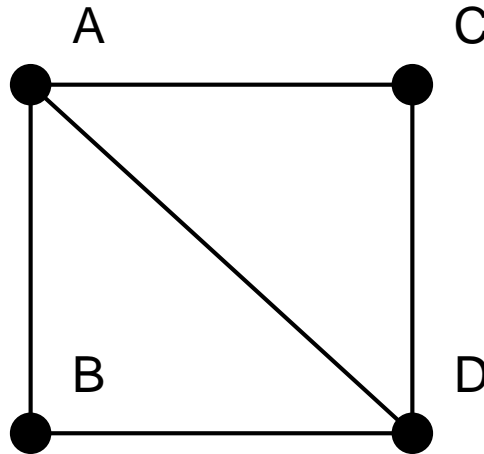
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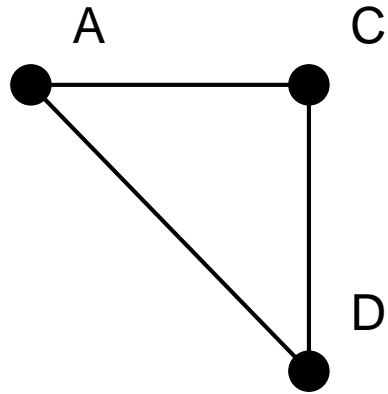
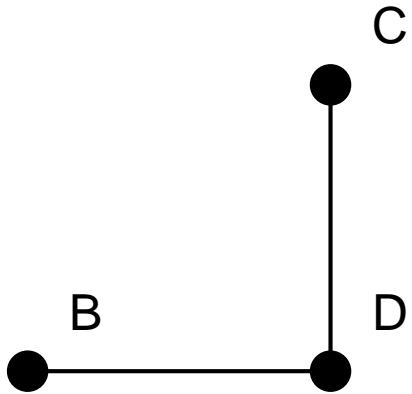
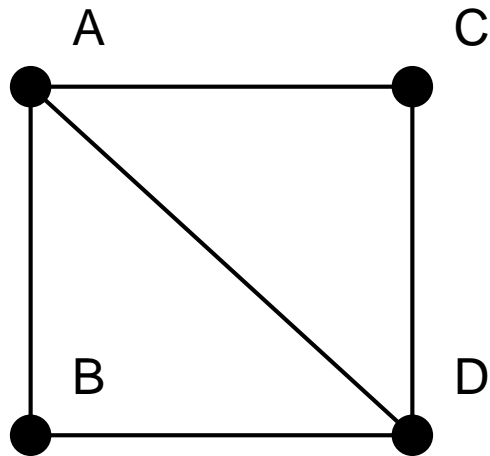
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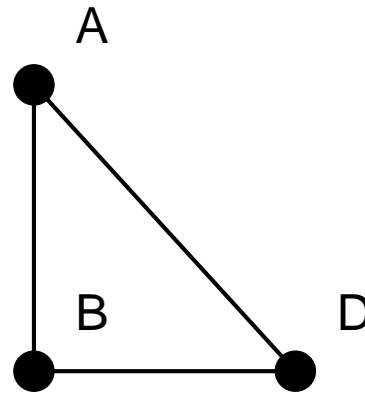
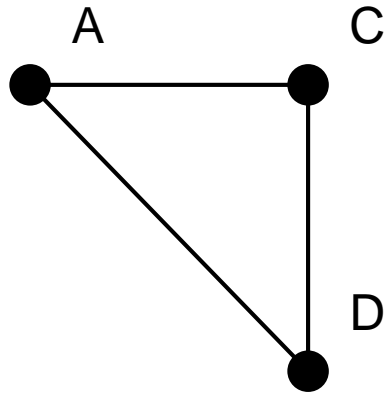
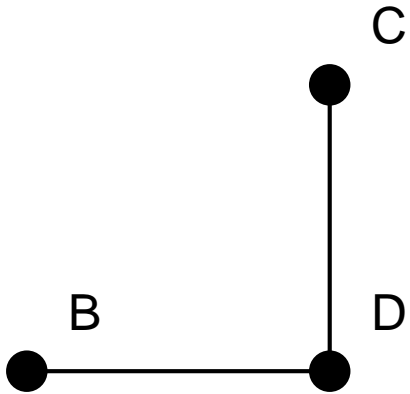
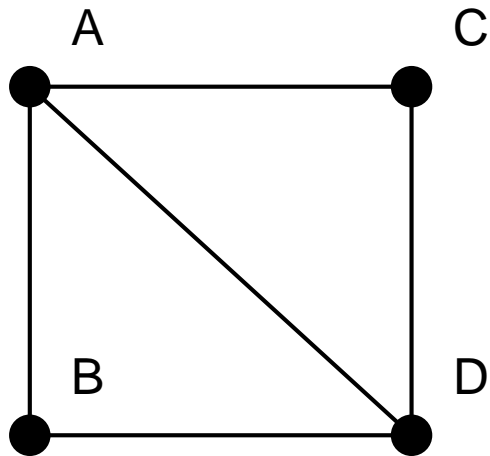
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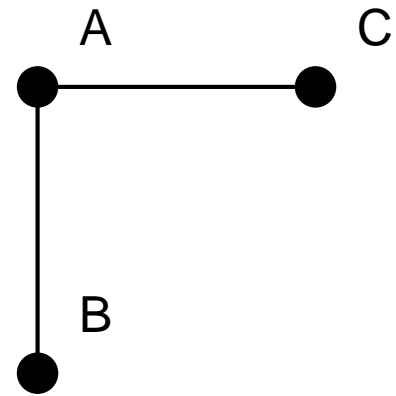
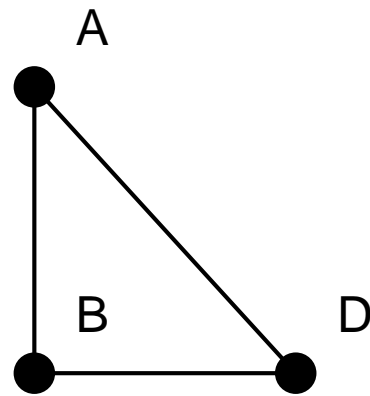
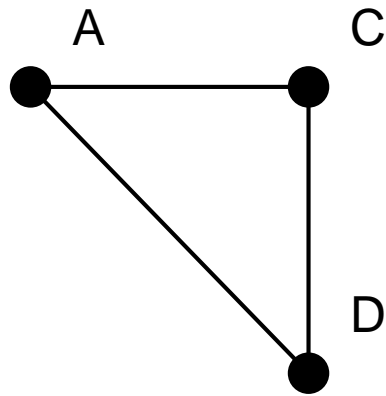
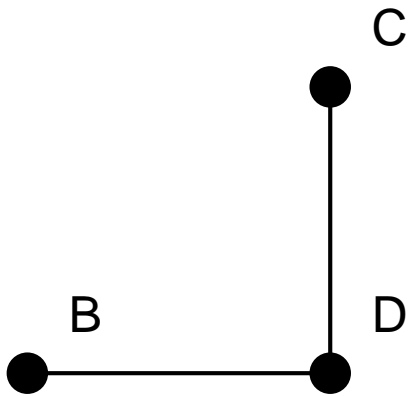
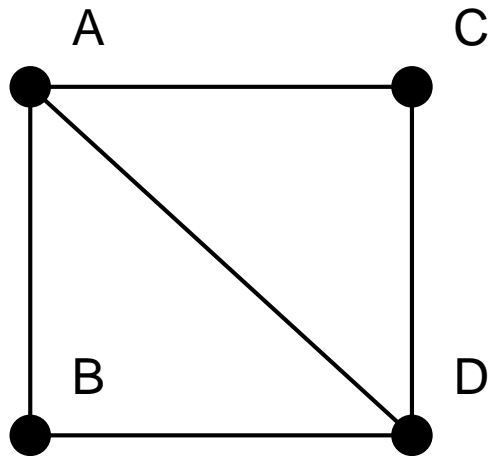
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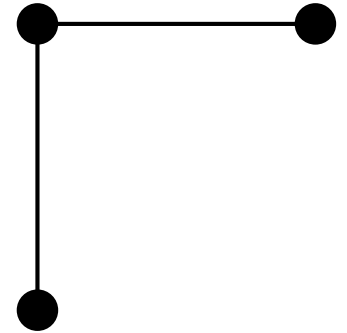
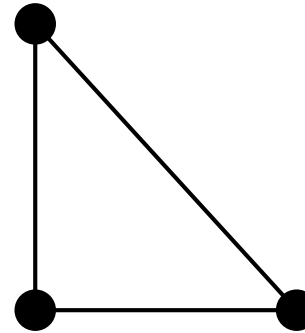
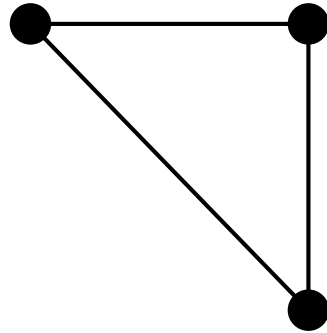
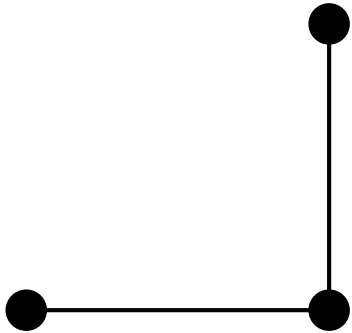
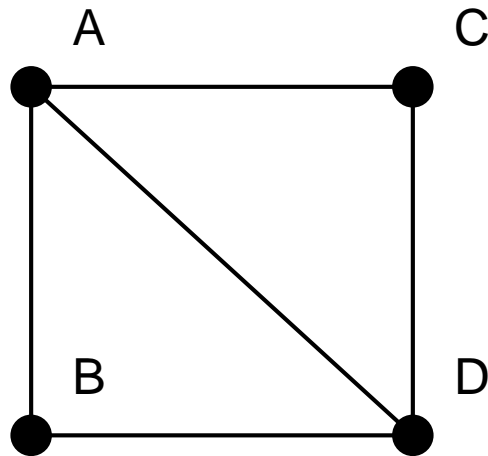
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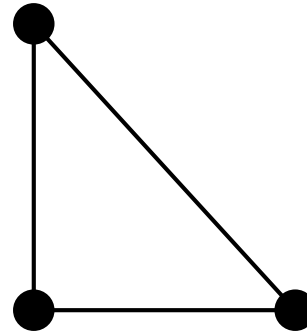
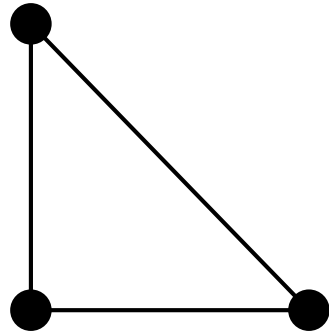
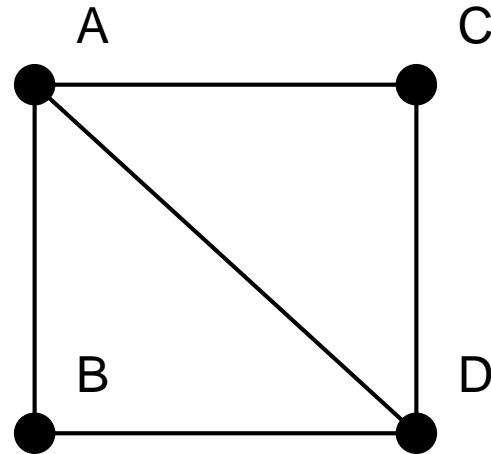
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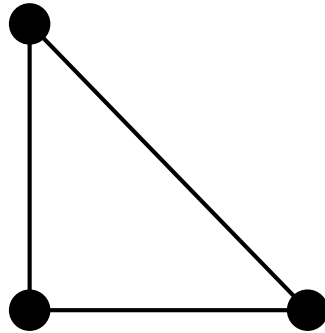
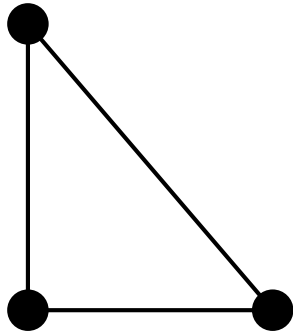
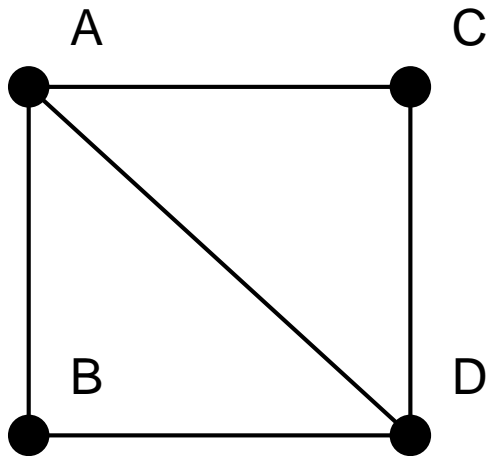
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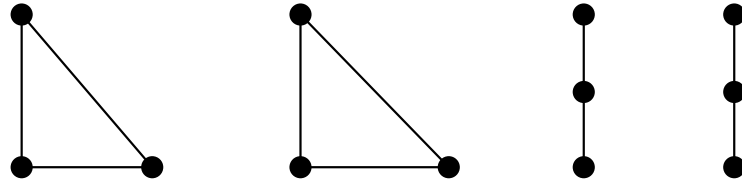
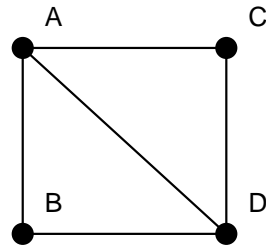
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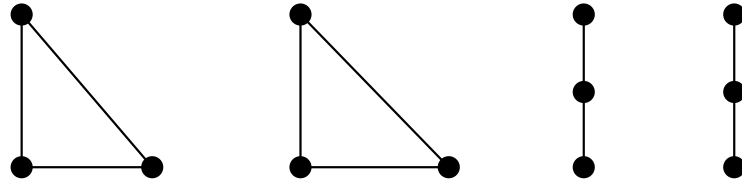
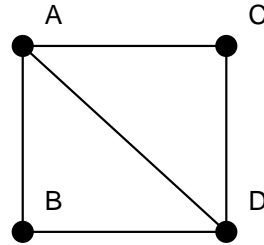
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- A graph is **reconstructible** if, given its deck, we can uniquely determine what the original graph was.

# Example

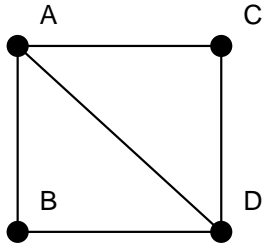
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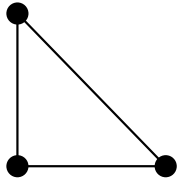
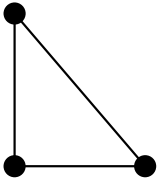
- A graph is **reconstructible** if, given its deck, we can uniquely determine what the original graph was.
- Equivalently,  $G$  is not reconstructible if and only if there is an  $H$  such that  $\mathcal{D}(G) = \mathcal{D}(H)$ .

# Example

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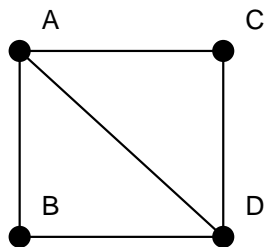


• This graph is reconstructible.

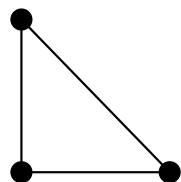
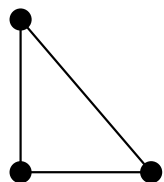


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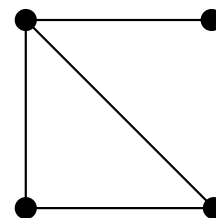
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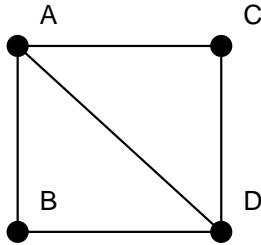


- The last three cards are also in the deck of

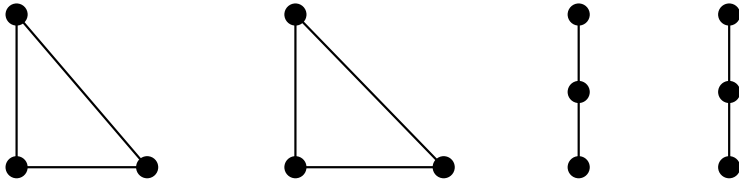


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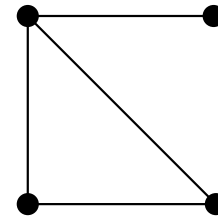
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- This graph is reconstructible.



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- (these can be checked by exhaustion).

# The Reconstruction Conjecture

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Every graph with at least 3 vertices is reconstructible.

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Every graph with at least 3 vertices is reconstructible.
- We need at least three vertices:



- Note: the reconstruction conjecture is false if and only if there exist two graphs with the same deck.

# Reconstructible Properties

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- Some properties of  $G$  are easily determined from  $\mathcal{D}(G)$ :

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  - $|V(G)| = \#$  of cards in  $\mathcal{D}(G)$ .
  - $|E(G)|$ : Each edge in  $G$  is missing from exactly two cards. Thus each edge is in exactly  $n - 2$  cards, and we have

$$|E(G)| = \frac{1}{n - 2} \sum_{v \in V(G)} |E(G - v)|.$$

# Further...

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**Lemma:**  $\forall v \in V(G)$ ,

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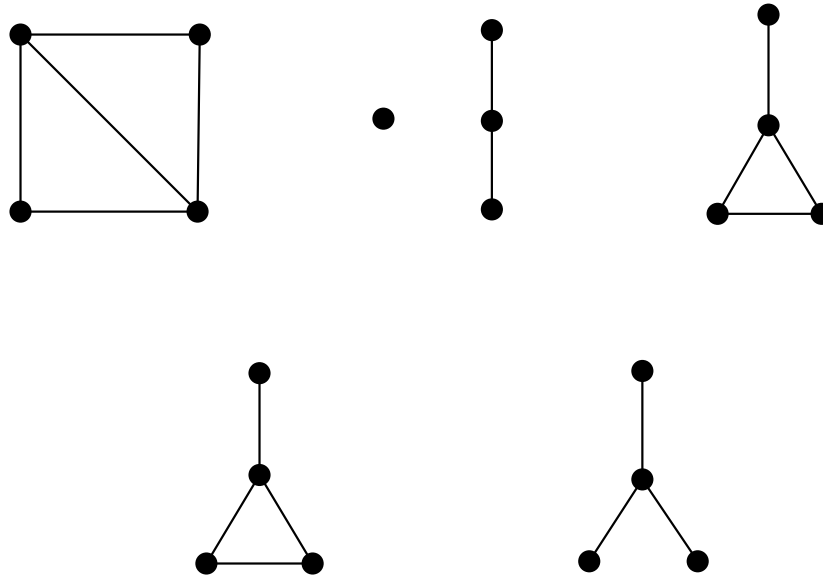
$$\deg(v) = |E(G)| - |E(G - v)|.$$

- Since  $|E(G)|$  can be determined from  $\mathcal{D}(G)$ , we can also determine the degree of  $v$  for all  $v \in V(G)$ .
- $\implies$  We can determine the whole *degree sequence* of  $G$ .

# Reconstruction Example

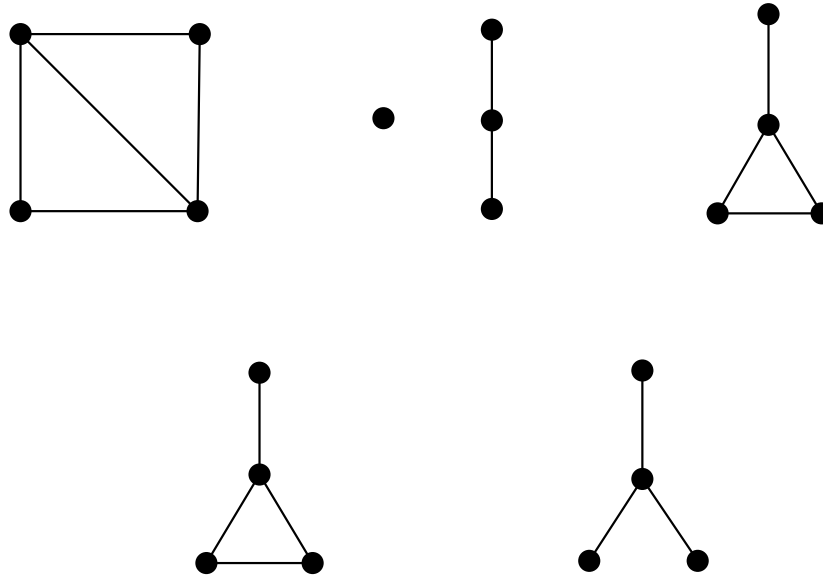
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Reconstruct, if possible, one or more graphs from the following deck:



# Reconstruction Example

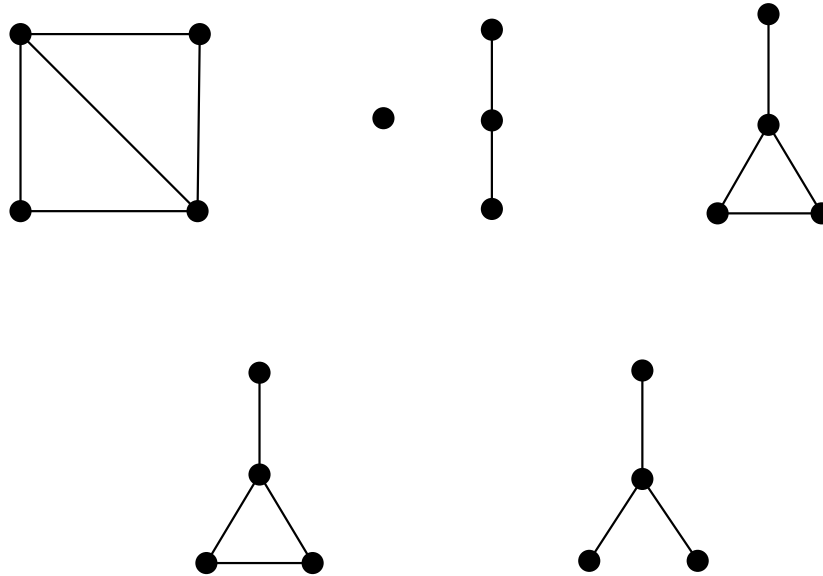
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$$|V(G)| = \# \text{ of cards.}$$

# Reconstruction Example

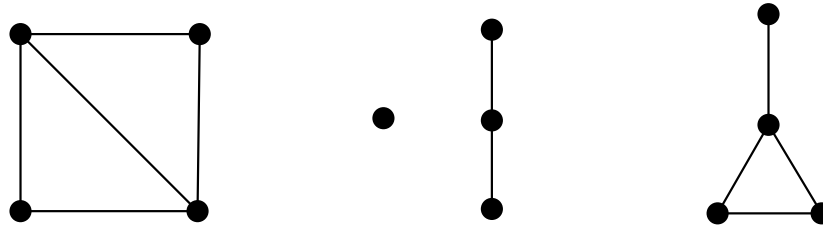
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$$|V(G)| = 5.$$

# Reconstruction Example

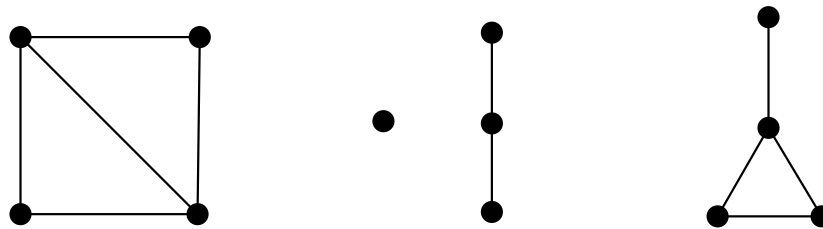
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# Reconstruction Example

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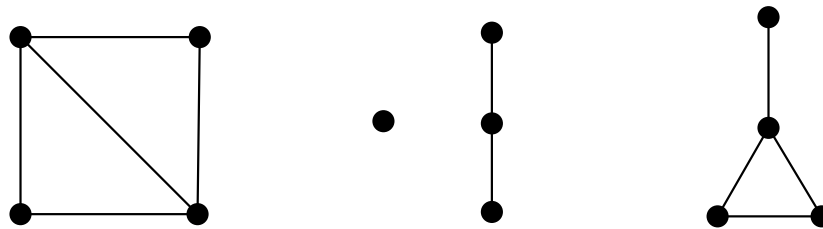


•  $|V(G)| = 5.$

$$|E(G)| = \frac{1}{|V(G)| - 2} \sum_{v \in V(G)} |E(G - v)|.$$

# Reconstruction Example

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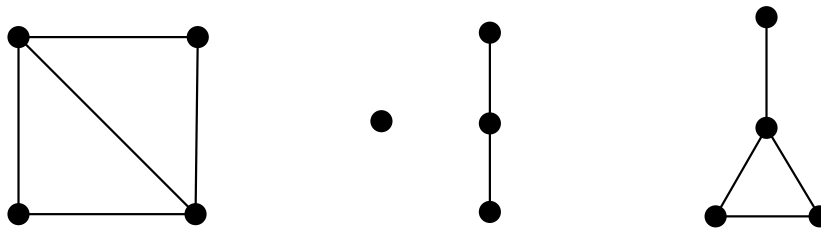


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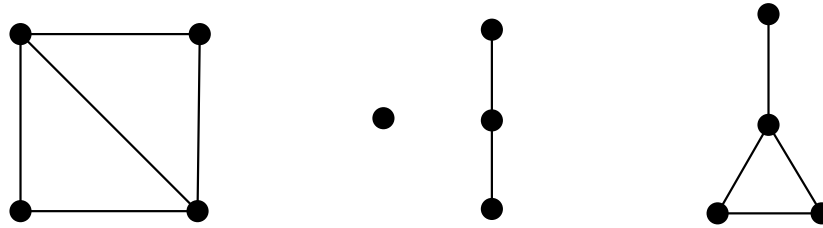


•  $|V(G)| = 5.$

$$|E(G)| = \frac{1}{\mathbf{5} - 2} \sum_{v \in V(G)} |E(G - v)|.$$

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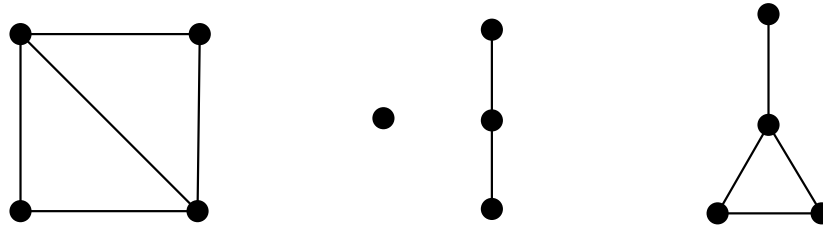


•  $|V(G)| = 5.$

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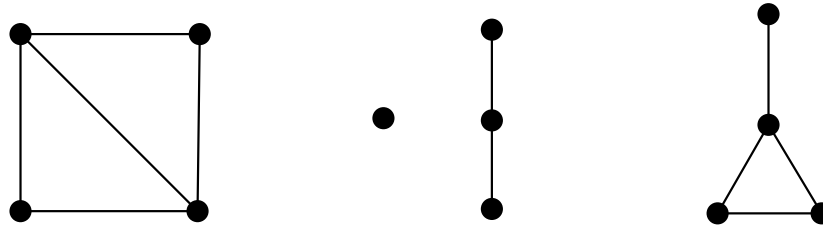


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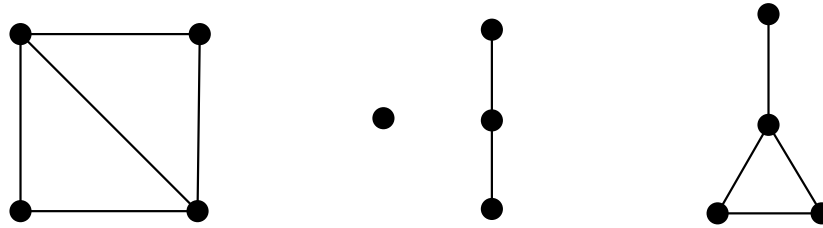


•  $|V(G)| = 5.$

$$|E(G)| = \frac{1}{3}(5 + 2 + 4 + 4 + 3).$$

# Reconstruction Example

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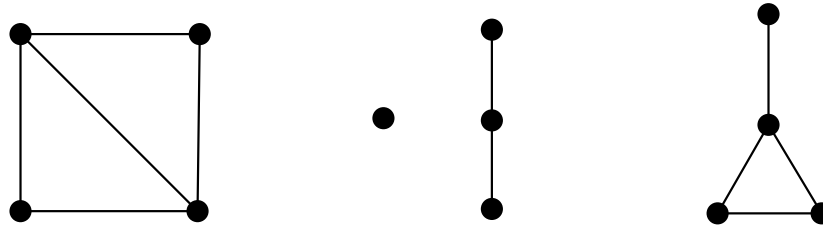


•  $|V(G)| = 5.$

$$|E(G)| = \frac{1}{3}(5 + 2 + 4 + 4 + 3) = 6.$$

# Reconstruction Example

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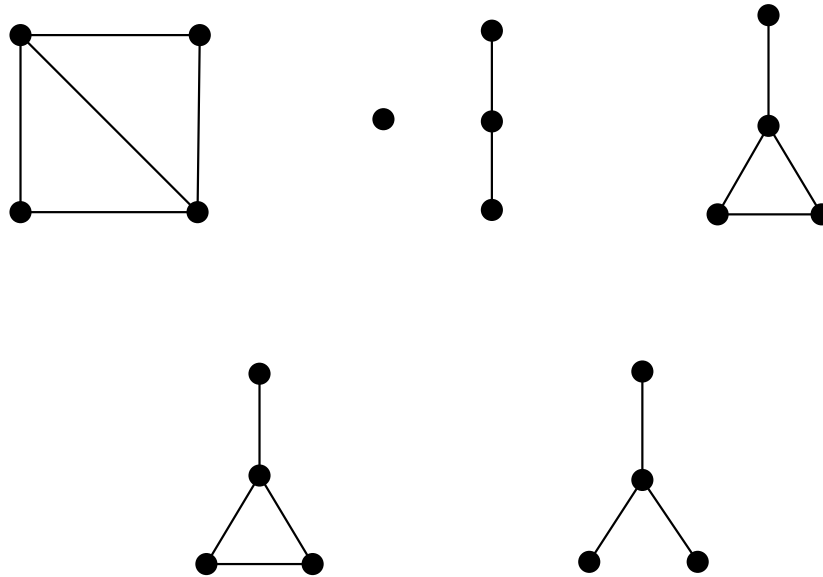


- $|V(G)| = 5.$

- $|E(G)| = 6.$

# Reconstruction Example

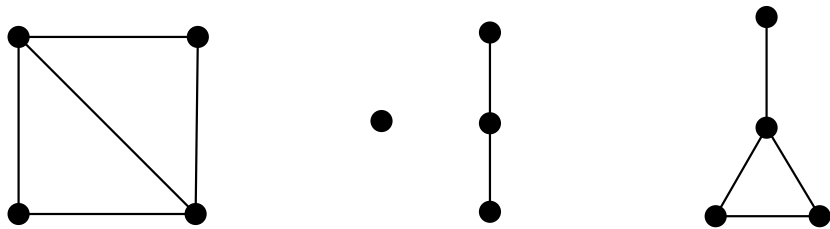
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- $|V(G)| = 5$ .
- $|E(G)| = 6$ .
- Thus the degrees of the missing vertices in order are 1, 4, 2, 2, 3.

# Reconstruction Example

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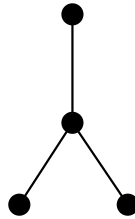
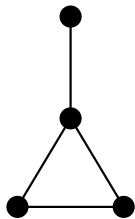
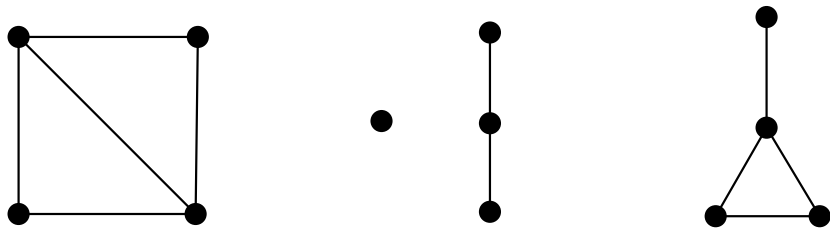


- Note the degree of the vertex deleted in the second graph is 4.



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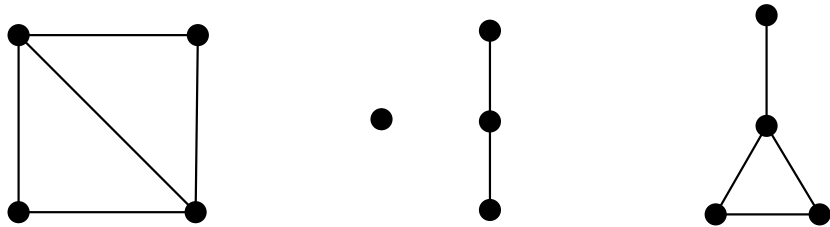
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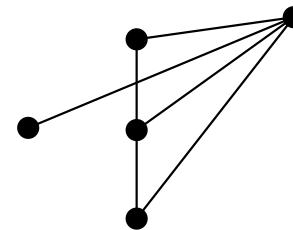
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- Thus it must have been connected to every other vertex:

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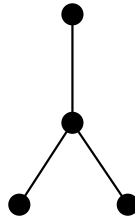
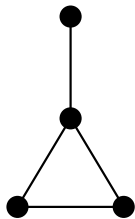
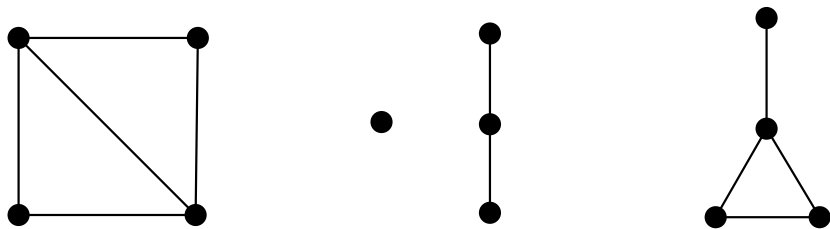


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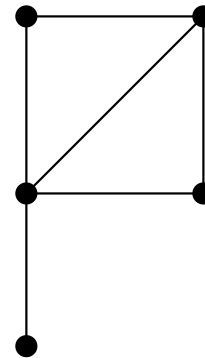


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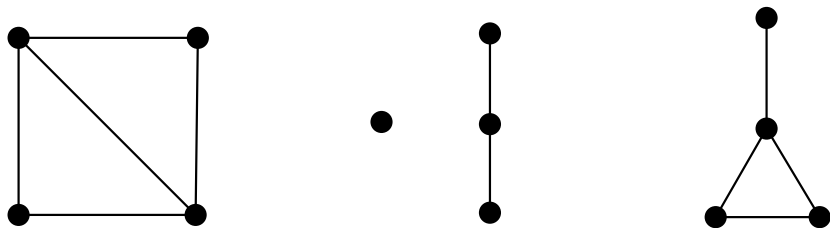


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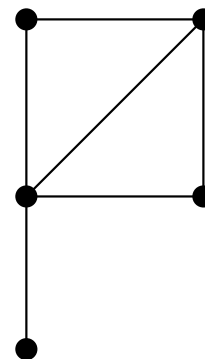


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**Determined Uniquely!**

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A graph is **regular** if all its vertices have the same degree.

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**Theorem**(Kelly, 1957) Regular graphs are reconstructible.

*Proof.*

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  - Let  $d$  be the degree of each vertex.
  - Add a vertex to any card.
  - Connect new vertex to each of the  $d$  vertices in the card with degree  $d - 1$ , and  $G$  is reconstructed uniquely.
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# Kelly's Lemma

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- **Kelly's Theorem:** For all  $F$  with less vertices than  $G$ ,

$$s(F, G) = \frac{1}{(|V(G)| - |V(F)|)} \sum_{v \in V(G)} s(F, G - v).$$

# Proof of Kelly's Theorem:

---

Double count ordered pairs of the form

$$(H, G - v)$$

where  $H \cong F$ , and  $H \subset G - v$ .

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Thus, we have that Kelly's Lemma implies that the number of edges is determined by the deck.

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 $s(F, G)$  **for any**  $F$ .
- Thus if there are two graphs with the same deck, they must have the same:
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  - number of edges.
  - number of subgraphs of EVERY type (same  $s(F, G)$  for all  $F$  with  $|V(F)| < |V(G)|$ ).

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- The conjecture has been confirmed after all for all graphs with less than 11 vertices ( 12.3 million graphs).
- However, there are good reasons to believe it false...

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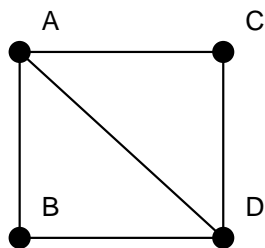
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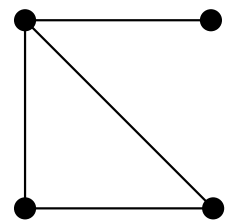
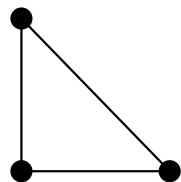
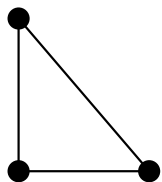
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- Almost all graphs have  $\exists rn(G) = 3$ .  
(Bollobas, 1990).

# Example

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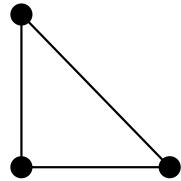
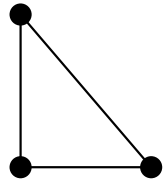
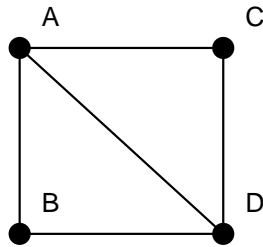


• The last three cards in this deck are as well in the deck of

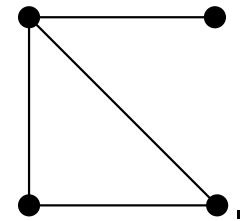


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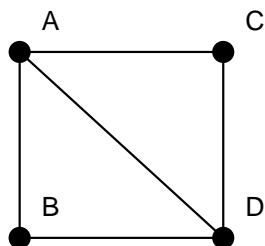
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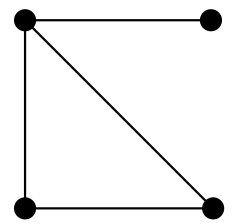
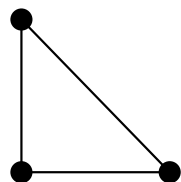
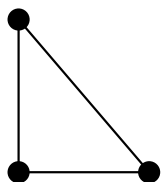
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- However, the first three graphs DO determine the graph  $G$  uniquely.
- Thus, for this  $G$ ,  
 $\exists rn(G) = 3, \forall rn(G) = 4.$

# Reconstruction Numbers

---

- Found for all graphs on at most 10 vertices:

| $\exists rn$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n = 9$ | $n = 10$   |
|--------------|---------|---------|---------|---------|---------|---------|---------|------------|
| 3            | 4       | 8       | 34      | 150     | 1044    | 12,334  | 274,666 | 12,005,156 |
| 4            |         | 3       |         | 4       |         | 8       |         | 6          |
| 5            |         |         |         | 2       |         | 2       | 2       | 4          |
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- Very hard to get any further: there are 1,018,997,864 graphs on 11 vertices!

# Other Related Problems

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  - The Graph Reconstruction Conjecture says  $f(1) = 2$ .

# References

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