An Introduction To Ramsey Theory for Graphs

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Abstract

A graph is a set of vertices with some pairs of vertices connected by edges. Graphs are used to model a number of phenomena, from biological and physical real life problems, to number theoretic and other mathematical problems. The study of partitioning substructures of graphs has been studied quite extensively. Ramsey theory is often described as the study of preservation of structure under partitioning. In this talk, I will survey some of the classic Ramsey theory results, and survey specifically some results from Ramsey theory on graphs.

Graph Theory

The study of Graphs ...

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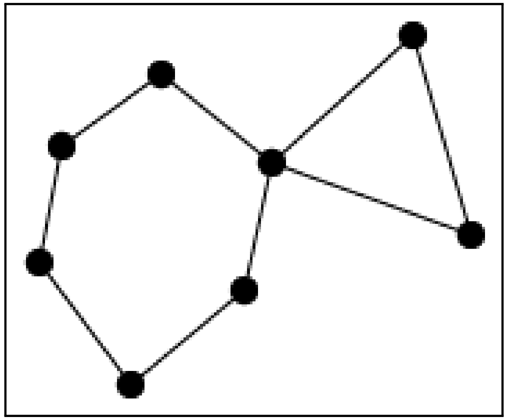
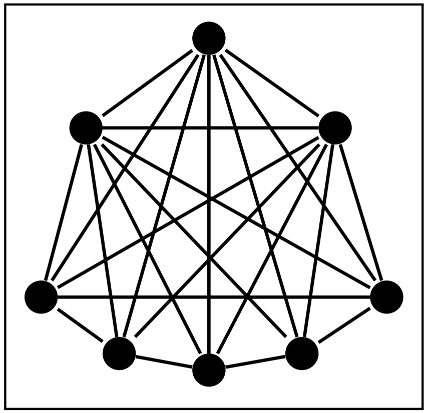
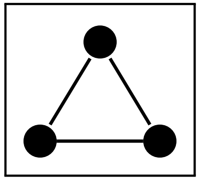
The study of Graphs ... so what's a graph?

Graph Theory







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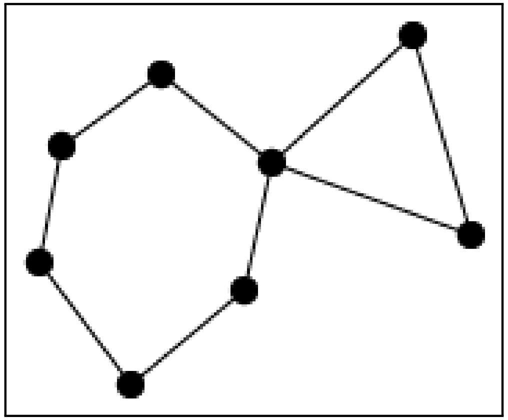

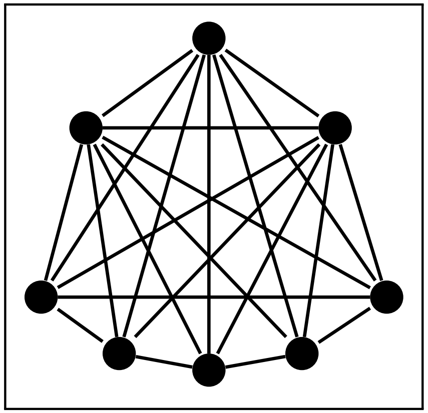

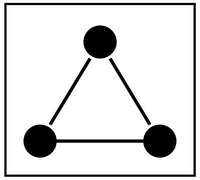

- We consider only the simplest types of graphs

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> Think of the pigeon hole principle - two pigeons in one hole is the "structure" we are guaranteed.

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  - (see overhead)

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- $\{1, \ldots, 9\} \to (\text{AP}_3)^{numbers}_2$

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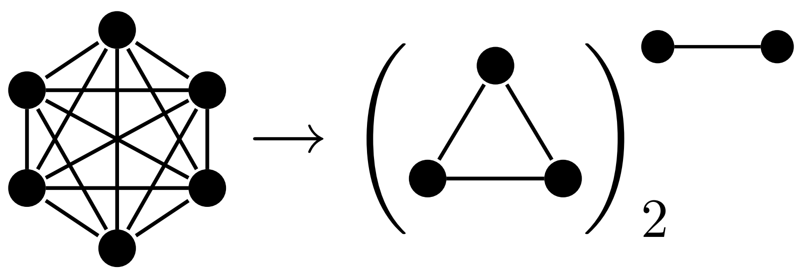

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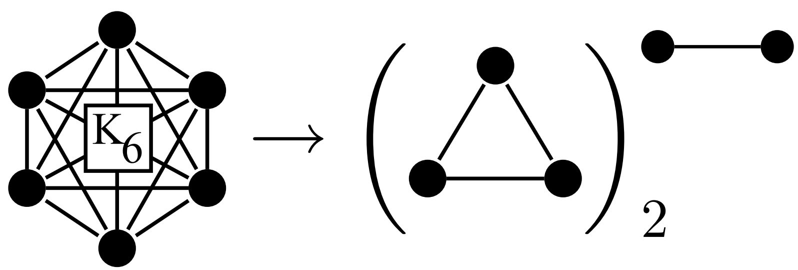

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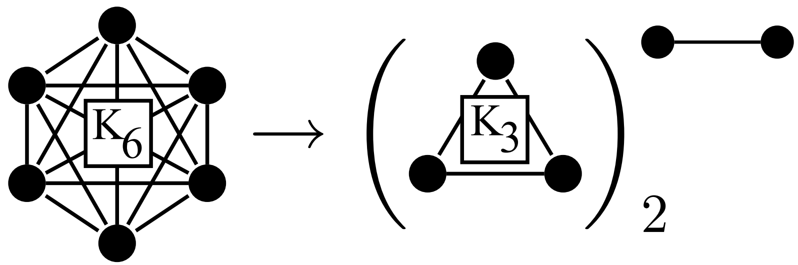

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- This shows that $R(3) \leq 6$. (Easy exercise to show $R(3) = 6$)

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- Ramsey numbers are hard to find (even the small values!)

Sparse Ramsey Graphs: K_k-free

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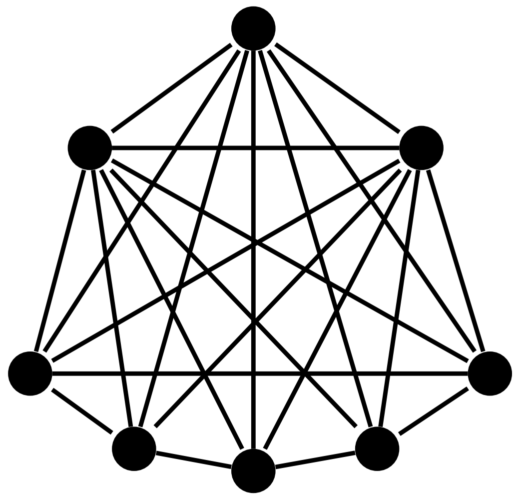

Found in 1968.

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YES! Found in 1999.
15 vertices suffices.
659 distinct graphs found.
This IS minimum.

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YES! Found in 1989. 3,000,000,000 vertices! Probabilistic (existence theorem only). Nothing better known!

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- So, the graphs don't even have to be very dense with edges at all!
- Other results have shown that we can find a graph that works with pretty much any property we want.

Conclusion

Thanks for coming!